Bayer-Macrì decomposition on Bridgeland moduli spaces over surfaces

by

Wanmin Liu

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This is to certify that I have examined the above doctor of philosophy thesis and have found that it is complete and satisfactory in all respects, and that any and all revisions required by the thesis examination committee have been made.

Prof. Wei-Ping Li, Supervisor

Prof. Yang Wang, Head of Department

Department of Mathematics

June 2015

To my beloved wife Jiewei, and our forthcoming baby

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Bayer-Macrì decomposition on Bridgeland moduli spaces over surfaces

Wanmin Liu

Department of Mathematics
The Hong Kong University of Science and Technology

Abstract

We find a decomposition formula of the local Bayer-Macrì map for the nef line bundle theory on the Bridgeland moduli space over surface and obtain the image of the local Bayer-Macrì map in its Néron-Severi group. The geometric meaning of the decomposition is given. Fix a base moduli space and assume that we can identify Néron-Severi groups of different birational models of the moduli space. We then have a global Bayer-Macrì map. By the decomposition formula, we obtain a precise correspondence between Bridgeland walls in stability manifold of the surface and Mori walls in pseudo-effective cone of divisors on the base moduli space. As one application, we solve a conjecture raised by Arcara, Bertram, Coskun, Huizenga (Adv. Math. 235(2013), 580-626) on the Hilbert scheme of points over projective plane. As another application, we find the ample cone of Hilbert scheme of points over Hirzebruch surface or a special elliptic surface.

Chapter 1

Introduction

1.1 Background

To understand the physical notion of Π-stability for Dirichlet branes [Dou02] in string theory, Bridgeland [Bri07] introduced stability conditions on triangulated categories. Let S be a smooth projective surface over $\mathbb{C}$ and $\mathrm{D^b}(S)$ be the bounded derived category of coherent sheaves on S. In mathematical aspect, since the triangulated category $\mathrm{D^b}(S)$ has more objects and morphisms than the abelian category $\mathrm{Coh}(S)$, the geometry of Bridgeland moduli space of complexes is richer than the geometry of the Gieseker moduli space of sheaves, and some hidden structures in classical geometry will be clear in the Bridgeland's setting.

Bridgeland [Bri07, Bri08] firstly constructed a family of stability conditions when S is a K3 or an abelian surface. Arcara and Bertram [AB13] extended the construction to any smooth projective surface. Denote such a stability condition by $\sigma_{\omega,\beta} = (Z_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$, which depends on a choice of two line bundles ω, β on S, with ω ample. The notation $\mathcal{A}_{\omega,\beta}$ denotes a heart of a t-structure of the derived category $\mathrm{D^b}(\mathrm{Coh}(S))$. The notation $Z_{\omega,\beta}$ is a group homomorphism from the Grothendieck group of $\mathrm{D^b}(S)$ to $\mathbb{C}$ which satisfies some assumptions. $\sigma_{\omega,\beta}$ is a *geometric* stability condition (see Remark 2.4.2). Fix the Chern characters $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$, and assume that they are of the Bogomolov type, i.e. $\mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2 \geq 0$. Bridgeland [Bri07, Bri08] and Toda [Tod08] showed that there is a wall-chamber structure for ch in the stability manifold $\mathrm{Stab}(S)$. Maciocia [Mac14] showed that the walls for geometric stability conditions are nested semicircles in a suitable parameter space. The non-geometric stability conditions are constructed by Toda [Tod13, Tod14].

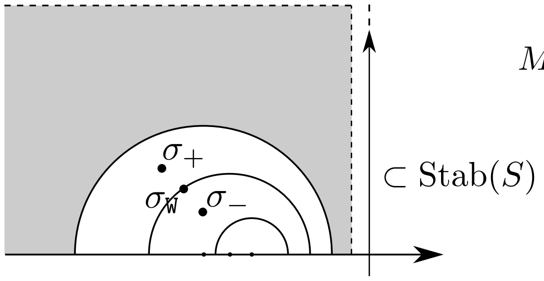
Assume $\mathrm{ch}_0 > 0$. Denote K_S the canonical divisor of the surface S. A Bridgeland stability condition $\sigma_{\omega,\beta}$ in the large volume limit (i.e. $\omega^2 \gg 0$) is the same as an α-twisted ω-Gieseker stability condition, where $\alpha := \beta - \frac{1}{2}\mathrm{K}_S$ (see Appendix A). So we obtain the isomorphic moduli spaces

$$M := M_{(\alpha,\omega)}(\mathrm{ch}) \cong M_{\sigma_{\omega,\beta}}(\mathrm{ch}), \text{ for } \omega^2 \gg 0,$$

where $M_{\sigma_{\omega,\beta}}(\mathrm{ch})$ denotes the Bridgeland moduli space of $\sigma_{\omega,\beta}$-semistable objects in $\mathrm{D^b}(S)$ with invariants ch, and $M_{(\alpha,\omega)}(\mathrm{ch})$ denotes the Gieseker moduli space of α-twisted ω-semistable sheaves in $\mathrm{Coh}(S)$ with the same invariants. Thus the classical wall-chamber structure for the twisted Gieseker stability conditions [EG95, FQ95, MW97] can be revisited as the Bridgeland wall-chamber structure in the large volume limit (see Table 4.1). Since there is a geometric invariant theory (GIT) construction of $M_{(\alpha,\omega)}(\mathrm{ch})$, the Bridgeland moduli space $M_{\sigma_{\omega,\beta}}(\mathrm{ch})$ is projective for $\omega^2 \gg 0$. However, the notion of Bridgeland stability condition is not from GIT. It is a challenging problem to show the projectivity of $M_{\sigma_{\omega,\beta}}(\mathrm{ch})$ for a general Bridgeland stability condition $\sigma_{\omega,\beta}$.

1.2 Motivations and ideas

There are two ways of viewing the birational geometry of $M = M_{(\alpha,\omega)}(\mathrm{ch})$. One way is the classical minimal model program (MMP) for M. In good cases (e.g. M is a Mori dream space), there is a *Mori wall-chamber structure* in the pseudo-effective cone $\overline{\mathrm{Eff}}(M)$ of divisors on M [HK00]. For a big divisor D on M, there is a rational map $M \dashrightarrow M(D) := \mathrm{Proj}(\mathrm{R}(M,D))$ if the section ring $\mathrm{R}(M,D) := \bigoplus_{m\geq 0} H^0(M, mD)$ is finitely generated. Then one obtains birational models $M(D)$ of M when D runs over the Mori wall-chamber structure. The other way is envisioned by Bridgeland [Bri08]. Note that $M \cong M_{\sigma_{\omega,\beta}}(\mathrm{ch})$ for $\sigma_{\omega,\beta}$ in the large volume limit. We then move $\sigma_{\omega,\beta}$ in the *Bridgeland wall-chamber structure* for ch in $\mathrm{Stab}(S)$. It is *expected* that there are natural contraction morphisms $M_{\sigma_\pm}(\mathrm{ch}) \to M_{\sigma_{\mathtt{W}}}(\mathrm{ch})$ when moving the stability conditions $\sigma_\pm$ in different adjacent chambers to $\sigma_{\mathtt{W}}$ in the wall. Set-theoretically it is *expected* that some strictly $\sigma_{\pm}$-stable objects become $\sigma_{\mathtt{W}}$-semistable, and the S-equivalent classes of $\sigma_{\mathtt{W}}$-semistable objects represent the contacted points in $M_{\sigma_{\mathtt{W}}}(\mathrm{ch})$.



$$\begin{tikzcd}
M_{\sigma_+}(\mathrm{ch}) \arrow[rr, dashed, "\text{birational}"] \arrow[dr] & & M_{\sigma_-}(\mathrm{ch}) \arrow[dl] \\
& M_{\sigma_{\mathtt{W}}}(\mathrm{ch}) &
\end{tikzcd}$$

Figure 1.1: Birational morphism via wall-crossing

Recently Bayer and Macrì [BM14a] constructed a nef determinant line bundle $\ell_{\sigma_{\omega,\beta}}$ on $M_{\sigma_{\omega,\beta}}(\mathrm{ch})$. This line bundle theory provides properties of the projectivity and birational geometry of $M_{\sigma_{\omega,\beta}}(\mathrm{ch})$. The line bundle $\ell_{\sigma_{\omega,\beta}}$ varies naturally via wall-crossing of stability conditions in the Bridgeland wall-chamber structure for ch. Assume that the Néron-Severi group of $M_{\sigma_{\omega,\beta}}(\mathrm{ch})$ can be identified with the Néron-Severi group of M. After the identification, the nef line bundle $\ell_{\sigma_{\omega,\beta}}$ on $M_{\sigma_{\omega,\beta}}(\mathrm{ch})$ is an effective line bundle on M. Bayer and Macrì established a correspondence from $\sigma_{\omega,\beta}$ in the Bridgeland wall-chamber structure of $\mathrm{Stab}(S)$ for ch, to $\ell_{\sigma_{\omega,\beta}}$ in the Mori wall-chamber structure of $\overline{\mathrm{Eff}}(M)$, which connects the two aspects of birational geometry of M.

The correspondence was also observed in some special cases by many people. Before Bayer and Macrì's theory, Arcara, Bertram, Coskun, Huizenga [ABCH13] conjectured that there is a precise correspondence between Bridgeland walls for $\mathrm{ch} = (1, 0, -n)$ in $\mathrm{Stab}(\mathbb{P}^2)$ and Mori walls in $\overline{\mathrm{Eff}}(M)$, for $M = \mathbb{P}^{2[n]}$ the Hilbert scheme of n-points on $\mathbb{P}^2$ (Figure 6.1). They verified the cases $n = 2, \cdots, 9$. The speculation was further extended to other rational surfaces by Bertram and Coskun [BC13]. Bayer and Macrì [BM14b] also established more concrete correspondence on K3 surfaces by using lattice theory in the case that the Mukai vector $v(\mathrm{ch})$ is primitive. The case when the Mukai vector $v(\mathrm{ch})$ is of O'Grady type was further studied by Meachan and Zhang [MZ14]. The case when the surface is an Enriques surface was studied by Nuer [Nue14].

It is interesting to give the precise correspondence between Bridgeland walls and Mori walls. This is the main purpose of the thesis. The key observation is to express the central charge in terms of Mukai bilinear form explicitly, by introducing the logarithm Todd class. In this way, we get an explicit Mukai vector $w_{\sigma_{\omega,\beta}}$ for computing the local Bayer-Macrì map. We find that such Mukai vector $w_{\sigma_{\omega,\beta}}$ has a very nice decomposition, and each term in the decomposition has clear geometric meaning. We call such decomposition as *Bayer-Macrì decomposition*. In particular, the image of the local Bayer-Macrì map is obtained, and generically, the rank of such image in Néron-Severi group is one more than the Picard number of the surface. This *partially answers* a classical question: *Can every line bundle on moduli space be written as determinant line bundle?* In addition, by assumption on the identification of Néron-Severi groups of different birational models, the precise correspondence between Bridgeland walls and Mori walls is obtained by using the Bayer-Macrì decomposition formula, as well as Maciocia's theorem on the structure of Bridgeland walls. Furthermore, it is believed that the geometry of a surface will reflect the geometry of moduli space over such surface. Our formula gives a direct link from the ample cone of the surface to the ample cone of the moduli space over such surface. A toy model is explored for this phenomenon.

1.3 Summary of main results

To state our results precisely, we introduce some notations. Let S be a smooth projective surface over $\mathbb{C}$. Let $\sigma_{\omega,\beta} = (Z_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$ be the Bridgeland stability condition constructed by Arcara and Bertram. Denote the Mukai bilinear form on S by $\langle\cdot,\cdot\rangle_S$ (see Section 2.3). We reformulate the central charge $Z_{\omega,\beta}$ in terms of Mukai bilinear form

$$Z_{\omega,\beta}(E) := -\int_S \mathrm{e}^{-(\beta+\sqrt{-1}\omega)}.\mathrm{ch}(E) = \langle \mho_Z, v(E)\rangle_S,$$

where

$$\mho_Z = \mathrm{e}^{\beta-\frac{3}{4}\mathrm{K}_S+\sqrt{-1}\omega+\frac{1}{24}\mathrm{ch}_2(S)}$$

is a complex Mukai vector (see Lemma 2.4.1). Write the complex number $Z_{\omega,\beta}(E) = \Re Z(E) + \sqrt{-1}\Im Z(E)$. Define the Bridgeland slope by $\mu_\sigma(E) := \frac{-\Re Z(E)}{\Im Z(E)}$. Fix Chern characters $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2) \in H^*(S,\mathbb{Q})$ and assume that $\mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2 \geq 0$. Denote $\mathbf{v} = v(\mathrm{ch}) := \mathrm{ch}.\sqrt{\mathrm{td}(S)}$ the Mukai vector of ch. Define $\mathbf{v}^\perp := \{w \in H^*_{\mathrm{alg}}(S,\mathbb{Q}) \otimes \mathbb{R} |\ \langle w, \mathbf{v}\rangle_S = 0\}$. Taking the complex conjugate of the number $\langle \mho_Z, \mathbf{v}\rangle_S$, and multiplying it to $\mho_Z$, we still get a complex Mukai vector. Then we define a real Mukai vector as its negative imaginary part:

$$w_{\sigma_{\omega,\beta}}(\mathrm{ch}) := -\Im\left(\overline{\langle \mho_Z, \mathbf{v}\rangle_S}\cdot \mho_Z\right).$$

We simply write it as $w_{\omega,\beta}$ or w_σ. This Mukai vector plays a crucial role in the computation of the local Bayer-Macrì map. If $\mathrm{ch}_0 \neq 0$, we define two special kinds of Mukai vectors:

$$\begin{aligned}\mathbf{w}(\mathrm{ch}) &:= \left(1, -\frac{3}{4}\mathrm{K}_S, -\frac{\mathrm{ch}_2}{\mathrm{ch}_0} - \frac{1}{2}\chi(\mathcal{O}_S) + \frac{11}{32}\mathrm{K}_S^2\right),\\ \mathbf{m}(L,\mathrm{ch}) &:= \left(0, L, (\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \frac{3}{4}\mathrm{K}_S).H\right), \text{ where } L \in \mathrm{NS}(S)_{\mathbb{R}},\end{aligned}$$

Define $\mathbf{u}(\mathrm{ch}) := \mathbf{w}(\mathrm{ch}) + \mathbf{m}(\frac{1}{2}\mathrm{K}_S, \mathrm{ch})$. So

$$\mathbf{u}(\mathrm{ch}) = \left(1, -\frac{1}{4}\mathrm{K}_S, -\frac{\mathrm{ch}_2}{\mathrm{ch}_0} + \frac{\mathrm{ch}_1.\mathrm{K}_S}{2\mathrm{ch}_0} - \frac{1}{2}\chi(\mathcal{O}_S) - \frac{1}{32}\mathrm{K}_S^2\right).$$

Denote $\mu_\sigma(\mathrm{ch}) := \mu_\sigma(E)$ if $\mathrm{ch} = \mathrm{ch}(E)$. If there is a flat family ([BM14a, Definition 3.1]) $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(M_\sigma(\mathrm{ch}) \times S)$ of σ-semistable objects with invariants ch parametrized by a proper algebraic space $M_\sigma(\mathrm{ch})$ of finite type over $\mathbb{C}$, we then denote the Mukai morphism by $\theta_{\sigma,\mathcal{E}}$ (see equation (3.1.3)), and denote

$$\begin{aligned}\widetilde{L} &:= -\theta_{\sigma,\mathcal{E}}(\mathbf{m}(L,\mathrm{ch})), \quad \mathcal{B}_0 := -\theta_{\sigma,\mathcal{E}}(\mathbf{u}(\mathrm{ch})),\\ \mathcal{B}_\alpha &:= \widetilde{\beta} - \theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch})) = \widetilde{\alpha} + \mathcal{B}_0. \quad (\text{Recall } \alpha := \beta - \frac{1}{2}\mathrm{K}_S.)\end{aligned}$$

We get the following decomposition formula of the Bayer-Macrì line bundle.

Theorem 1.3.1. *(Bayer-Macrì decomposition for objects supported in dimension 2, Theorems 5.2.4 and 5.4.4.)*

(a). If $\mathrm{ch}_0 \neq 0$ and $\Im Z(\mathrm{ch}) > 0$, then $w_{\omega,\beta}$ has a decomposition (up to a positive scalar):

$$\begin{aligned}w_{\omega,\beta}(\mathrm{ch}) &\overset{\mathbb{R}_+}{=\!=} \mu_\sigma(\mathrm{ch})\mathbf{m}(\omega,\mathrm{ch}) + \mathbf{m}(\beta,\mathrm{ch}) + \mathbf{w}(\mathrm{ch}) &(1.3.1)\\ &= \mu_\sigma(\mathrm{ch})\mathbf{m}(\omega,\mathrm{ch}) + \mathbf{m}(\alpha,\mathrm{ch}) + \mathbf{u}(\mathrm{ch}), &(1.3.2)\end{aligned}$$

where $\mathbf{m}(\omega,\mathrm{ch})$, $\mathbf{m}(\beta,\mathrm{ch})$, $\mathbf{m}(\alpha,\mathrm{ch})$, $\mathbf{w}(\mathrm{ch})$, $\mathbf{u}(\mathrm{ch}) \in \mathbf{v}^\perp$.

(b). Assume that (i) $\mathrm{ch}_0 > 0$; (ii) there is a flat family $\mathcal{E}$; (iii) $M_{(\alpha,\omega)}(\mathrm{ch})$ is irreducible; (iv) $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$; (v) $U_\omega(\mathrm{ch})$ is of positive dimension; (vi)

we can identify the Néron-Severi groups as Section 3.4. Then the Bayer-Macrì line bundle class is

$$\ell_{\sigma_{\omega,\beta}} = -\mu_\sigma(\mathrm{ch})\widetilde{\omega} - \mathcal{B}_\alpha = -\mu_\sigma(\mathrm{ch})\widetilde{\omega} - \widetilde{\alpha} - \mathcal{B}_0. \tag{1.3.3}$$

The line bundle class $\widetilde{\omega}$ induces the Gieseker-Uhlenbeck (GU) morphism from the α-twisted ω-semistable Gieseker moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ to the Uhlenbeck space $U_\omega(\mathrm{ch})$.

(c). If $\sigma_{\omega,\beta}$ is a generic stability condition (i.e. not in a wall for ch*), then the image of the Bayer-Macrì map in $\mathrm{NS}(M_{\sigma_{\omega,\beta}}(\mathrm{ch}))$ is of rank one more than the Picard number of the surface.*

(d). If $\mathrm{ch}_0 = 2$, then the divisor $\mathcal{B}_\alpha = \widetilde{\alpha} + \mathcal{B}_0$ is the α-twisted boundary divisor of the induced GU morphism.

Assumptions in Theorem 1.3.1 are rather technical. An example is given when the surface is K3 in Theorem 1.3.4. To state the next result, we use Maciocia's notations. The details are given in Section 2.5. Fix a triple data (H, γ, u), where H is an ample divisor on S, $\gamma \in H^\perp$, (i.e. γ is another divisor on S, and the intersection of the two divisors vanishes $H.\gamma = 0$), and u is a real number. Take Maciocia's coordinate $\omega = tH$ and $\beta = sH + u\gamma$. Let ch' be the potential destabilizing Chern characters. Then potential (Bridgeland) walls $W(\mathrm{ch}, \mathrm{ch}')$ are nested semicircles with centers $(C(\mathrm{ch}, \mathrm{ch}'), 0)$ and radius $R = \sqrt{C(\mathrm{ch}, \mathrm{ch}')^2 + D(\mathrm{ch}, \mathrm{ch}')}$ in the (s,t)-half-plane $\Pi_{(H,\gamma,u)}$ $(t > 0)$, where $C(\mathrm{ch}, \mathrm{ch}')$ is defined by equation (2.5.4), and $D(\mathrm{ch}, \mathrm{ch}')$ is defined by equation (2.5.5). A potential wall $W(\mathrm{ch}, \mathrm{ch}')$ is a *Bridgeland wall* if there is a $\sigma \in \mathrm{Stab}(S)$ and objects E, $F \in \mathcal{A}_\sigma$ such that $\mathrm{ch}(E) = \mathrm{ch}$, $\mathrm{ch}(F) = \mathrm{ch}'$ and $\mu_\sigma(E) = \mu_\sigma(F)$.

Theorem 1.3.2. *(The Bridgeland-Mori correspondence for objects supported in dimension* 2 *and geometric Bridgeland stability conditions, Theorem 6.1.1.) Assume that (i) $\mathrm{ch}_0 > 0$; (ii) there is a flat family $\mathcal{E}$; (iii) we can identify the Néron-Severi groups as Section 3.4. Then there is a correspondence from the Bridgeland wall $W(\mathrm{ch}, \mathrm{ch}')$ on the half-plane $\Pi_{(H,\gamma,u)}$ with center $C(\mathrm{ch}, \mathrm{ch}')$ to the nef line bundle on the moduli space $M_\sigma(\mathrm{ch})$:*

$$\ell_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} = -C(\mathrm{ch}, \mathrm{ch}')\widetilde{H} - u\widetilde{\gamma} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0. \tag{1.3.4}$$

The above assumptions hold if $M = M_{(\alpha,\omega)}(\mathrm{ch})$ is a Mori dream space (Corollary 6.1.4). As an application, we solve a problem raised by Arcara, Bertram, Coskun, Huizenga [ABCH13] on the Hilbert scheme of n-points over projective plane in Corollary 6.2.2.

Above results are for objects supported in dimension 2, i.e. $\mathrm{ch}_0 \neq 0$. Similar results for objects supported in dimension 1 are given as follows. Assume $\mathrm{ch} = (0, \mathrm{ch}_1, \mathrm{ch}_2)$ with $\mathrm{ch}_1.H > 0$ for a choice of Maciocia's coordinate (H, γ, u). Then the center $(C, 0)$ is fixed, where $C = \frac{z+duy_2}{gy_1}$ (Theorem 2.5.2). $D(\mathrm{ch}, \mathrm{ch}')$ is computed by equation (2.5.7). Define the Mukai vector

$$\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}) := \left(1, CH + u\gamma - \frac{3}{4}\mathrm{K}_S, -\frac{3}{4}\mathrm{K}_S.[CH + u\gamma] - \frac{1}{2}\chi(\mathcal{O}_S) + \frac{11}{32}\mathrm{K}_S^2\right),$$

which is independent of the potential destabilizing Chern characters ch'. Denote

$$\mathcal{S} := \theta_{\sigma,\mathcal{E}}(0,0,-1), \quad \text{and} \quad \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}) := \theta_{\sigma,\mathcal{E}}(\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch})).$$

Theorem 1.3.3. *(Bayer-Macrì decomposition for objects supported in dimension 1, Theorem 5.2.1 and Theorem 5.4.1.) Assume* $\mathrm{ch} = (0, \mathrm{ch}_1, \mathrm{ch}_2)$ *with* $\mathrm{ch}_1.H > 0$ *for a choice of Maciocia's coordinate* (H, γ, u)*.*

(a). Then $w_{\sigma\in W(\mathrm{ch},\mathrm{ch}')}$ *has a decomposition:*

$$w_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} = \left(\frac{g}{2}D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2}u^2\right)(0, 0, -1) + \mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}), \tag{1.3.5}$$

where $(0, 0, -1)$, $\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}) \in \mathbf{v}^\perp$*. Moreover* $r = \mathrm{ch}'_0 \neq 0$ *and the coefficient of* $(0, 0, -1)$ *is expressed in terms of potential destabilizing Chern characters* $\mathrm{ch}' = (r, c_1H + c_2\gamma + \delta', \chi)$*:*

$$\frac{g}{2}D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2}u^2 = \frac{\chi - gCc_1 + udc_2}{r}. \tag{1.3.6}$$

(b). Assume (i) there is a flat family $\mathcal{E}$*; (ii) we can identify the Néron-Severi groups as Section 3.4. Then there is a correspondence from the Bridgeland wall on the* (s, t)*-half plane to the nef line bundle on the moduli space* $M_\sigma(\mathrm{ch})$*:*

$$\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} = \left(\frac{g}{2}D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2}u^2\right)\mathcal{S} + \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}). \tag{1.3.7}$$

The line bundle $\mathcal{S}$ is conjectured to induce the support morphism (Conjecture 4.6.3).

In the special case that S is a smooth K3 surface, we use the central charge $\hat{Z}_{\omega,\beta}(E) := -\int_S \mathrm{e}^{-(\beta+\sqrt{-1}\omega)}.\mathrm{ch}(E)\sqrt{\mathrm{td}(S)}$. Bridgeland [Bri08, Lemma 6.2] showed that if $\hat{Z}_{\omega,\beta}(F) \notin \mathbb{R}_{\leq 0}$ for all spherical sheaves $F \in \mathrm{Coh}(S)$, then $\hat{\sigma}_{\omega,\beta} := (\hat{Z}_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$ is a Bridgeland stability condition. Moreover the stability condition $\hat{\sigma}_{\omega,\beta}$ is reduced (Remark 2.4.3), which is the *Hodge theoretic restrictions* on the central charge [Bri09, BB13]. The potential walls $\hat{W}(\mathrm{ch}, \mathrm{ch}')$ are given by semi-circles $t^2 + (s-C)^2 = C^2 + D + \frac{2}{g}$, where C and D are defined in Theorem 2.5.2. There is a similar notation for the Bridgeland slope $\mu_{\hat{\sigma}_{\omega,\beta}}$ with respect to $\hat{\sigma}_{\omega,\beta}$. The following theorem is a specialization of Theorem 1.3.1, Theorem 1.3.2 and Theorem 1.3.3 (but using $\hat{Z}_{\omega,\beta}$ instead).

Theorem 1.3.4. *(Bayer-Macrì decomposition on K3 surfaces, Theorem B.2.1.) Let S be a smooth projective K3 surface and $\mathbf{v} = v(\mathrm{ch}) \in H^*_{\mathrm{alg}}(S, \mathbb{Z})$ be a primitive class with $\langle \mathbf{v}, \mathbf{v} \rangle_S > 0$. Assume that $\hat{Z}_{\omega,\beta}(F) \notin \mathbb{R}_{\leq 0}$ for all spherical sheaves $F \in \mathrm{Coh}(S)$.*

- *The case $\mathrm{ch} = (0, \mathrm{ch}_1, \mathrm{ch}_2)$ with $\mathrm{ch}_1.H > 0$ for a choice of Maciocia's coordinate (H, γ, u). Then $\hat{w}_{\omega,\beta}$ has a decomposition*

$$\hat{w}_{\sigma\in\hat{W}(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=} \left(\frac{g}{2}D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2}u^2\right)(0,0,-1) + \mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}), \tag{1.3.8}$$

  and the coefficient before $(0, 0, -1)$ is given by equation (5.2.2). Moreover, the Bayer-Macrì line bundle has a decomposition

$$\ell_{\hat{\sigma}\in\hat{W}(\mathrm{ch},\mathrm{ch}')} = \left(\frac{g}{2}D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2}u^2\right)\mathcal{S} + \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}), \tag{1.3.9}$$

  where $\mathcal{S}$ induces the support morphism.

- *The case $\mathrm{ch}_0 > 0$. Assume further $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$. Then*

$$\hat{w}_{\omega,\beta} = \mu_{\hat{\sigma}_{\omega,\beta}}(\mathrm{ch})\mathbf{m}(\omega, \mathrm{ch}) + \mathbf{m}(\beta, \mathrm{ch}) + \mathbf{w}(\mathrm{ch}). \tag{1.3.10}$$

  Moreover, the Bayer-Macrì line bundle has a decomposition

$$\ell_{\hat{w}_{\omega,\beta}} = (-\mu_{\hat{\sigma}_{\omega,\beta}}(\mathrm{ch}))\widetilde{\omega} - \widetilde{\beta} - \mathcal{B}_0. \tag{1.3.11}$$

  Fix a Maciocia's coordinate (H, γ, u). There is a Bridgeland-Mori correspondence, which maps a Bridgeland wall $\hat{W}(\mathrm{ch}, \mathrm{ch}')$ with center $(C, 0)$ in the plane $\Pi_{(H,\gamma,u)}$ to the line bundle on the Mori wall

$$\ell_{\hat{\sigma}\in\hat{W}(\mathrm{ch},\mathrm{ch}')} = -C\widetilde{H} - u\widetilde{\gamma} - \mathcal{B}_0. \tag{1.3.12}$$

  The line bundle $\widetilde{\omega}$ (or $\widetilde{H}$) induces the Gieseker-Uhlenbeck morphism.

1.4 Outline of the paper

Chapter 2 is a brief review of the notion of Bridgeland stability conditions. The key ingredient is an explicit formula (2.4.1) to express the central charge in terms of the Mukai bilinear form. Chapter 3 is a brief review of Bayer and Macrì's line bundle theory on Bridgeland moduli spaces. Chapter 4 serves as the geometric background for the decomposition formulas in Chapter 5. Some terminologies are summarized in Figure 4.1. We prove Theorem 1.3.3 and Theorem 1.3.1 in Chapter 5, and prove Theorem 1.3.2 in Chapter 6. As one application, we give a positive answer to Arcara-Bertram-Coskun-Huizenga's conjecture in Corollary 6.2.2 which concerns the Hilbert scheme $\mathbb{P}^{2[n]}$ of n-points over $\mathbb{P}^2$. As another application, we study the Hilbert scheme $S^{[n]}$ of n-points over a particular surface S in Section 6.3, where S is either a $\mathbb{P}^1$-fibered or an elliptic-fibered surface over $\mathbb{P}^1$ with a section. The nef cone of $S^{[n]}$ is given in Theorem 6.3.2. Some background for Chapter 4 is given as Appendix A. Some parallel computations by using $\hat{Z}_{\omega,\beta}$ are given in Appendix B. We prove Theorem 1.3.4 for K3 surfaces in Appendix B.2. The notions on divisors, cones and Mori dream spaces are recalled in Appendix C.

1.5 Notations and Terminologies

We work throughout over the complex numbers $\mathbb{C}$. A *scheme* is a separated algebraic scheme of finite type over $\mathbb{C}$ [Har77]. A *variety* is a reduced and irreducible scheme. Let X be an irreducible projective scheme over $\mathbb{C}$.

Derived dual Let us recall some properties of the derived dual. Let S be a smooth projective surface over $\mathbb{C}$. For an object $E \in \mathrm{D^b}(S)$, define the derived dual object as $E^\vee := R\mathcal{H}om(E, \mathcal{O}_S) \in \mathrm{D^b}(S)$. We have $E^{\vee\vee} = E$, $(E[-1])^\vee = E^\vee[1]$, $(E[1])^\vee = E^\vee[-1]$. If $\mathrm{ch}(E) = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ then $\mathrm{ch}(E^\vee) = (\mathrm{ch}_0, -\mathrm{ch}_1, \mathrm{ch}_2) =: (\mathrm{ch}(E))^*$. If F is a sheaf, denote $F^* := \mathcal{H}om(F, \mathcal{O}_S)$ the dual sheaf. Define the duality functor $\Phi(\cdot) := (\cdot)^\vee[1]$.

Moduli spaces Fix the Chern characters $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2) \in H^*(S, \mathbb{Q})$. Choose $\omega, \beta \in N^1(S)$ with ω ample. Denote $\alpha := \beta - \frac{1}{2}\mathrm{K}_S$.

μ_ω:	Mumford slope of a torsion free sheaf E, $\mu_\omega(E) := \frac{\mathrm{ch}_1(E).\omega}{\mathrm{ch}_0}$.
μ_σ:	Bridgeland slope of an object E, $\mu_\sigma(E) := -\frac{\Re(Z(E))}{\Im(Z(E))}$.
(α,ω)-s.s.:	α-twisted ω-Gieseker semistability, Definition A.0.2.
$M_{\sigma_{\omega,\beta}}(\mathrm{ch})$:	moduli space of $\sigma_{\omega,\beta}$-s.s. objects E with $\mathrm{ch}(E) = \mathrm{ch}$.
$M_{(\alpha,\omega)}(\mathrm{ch})$:	moduli space of (α,ω)-s.s. sheaves E with $\mathrm{ch}(E) = \mathrm{ch}$.
$M_\omega(\mathrm{ch})$:	moduli space of (0-twisted) ω-s.s. sheaves E with $\mathrm{ch}(E) = \mathrm{ch}$.
$M^{\mathrm{lf}}_\omega(\mathrm{ch})$:	moduli space of μ_ω-s.s. locally free sheaves with invariant ch.
$U_\omega(\mathrm{ch})$:	Uhlenbeck compactification of $M^{\mathrm{lf}}_\omega(\mathrm{ch})$.

Maciocia's coordinate Fix a Maciocia's coordinate (H,γ,u) (Definition 2.5.1) and consider walls in the (s,t)-half plane $\Pi_{(H,\gamma,u)}$:

$$H \in \mathrm{Amp}(S),\ \gamma \in N^1(S) \cap H^\perp,\ \text{i.e. } H.\gamma = 0,\ g := H.H,\ d := -\gamma.\gamma, \tag{1.5.1}$$

$$\omega := tH\ (t > 0),\ \beta := sH + u\gamma\ (s \in \mathbb{R},\ u \in \mathbb{R} \text{ is fixed}), \tag{1.5.2}$$

$$\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2) =: (x, y_1H + y_2\gamma + \delta, z),\ \delta \in \{H,\gamma\}^\perp \tag{1.5.3}$$

$$\mathrm{ch}' = (\mathrm{ch}'_0, \mathrm{ch}'_1, \mathrm{ch}'_2) =: (r, c_1H + c_2\gamma + \delta', \chi),\ \delta' \in \{H,\gamma\}^\perp. \tag{1.5.4}$$

Miscellaneous For a complex number $z \in \mathbb{C}$, the real part and imaginary part of z are denoted by $\Re z$ and $\Im z$ respectively.

Let E be a semistable sheaf of dimension d. A *Jordan-Hölder filtration* of E is a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E, \tag{1.5.5}$$

such that the factors $\mathrm{gr}_i(E) := E_i/E_{i-1}$ are stable with reduced Hilbert polynomial $p(E)$. Jordan-Hölder filtration always exist [HL10, Proposition 1.5.2]. Up to isomorphism, the sheaf $\mathrm{gr}(E) := \oplus_i \mathrm{gr}_i(E)$ does not depend on the choice of the Jordan-Hölder filtration. Two semistable sheaves E and F with the same reduced Hilbert polynomial are called *S-equivalent* if $\mathrm{gr}(E) \cong \mathrm{gr}(F)$.

Chapter 2

Bridgeland stability conditions

Let S be a smooth projective surface over $\mathbb{C}$ and $\mathrm{D^b}(S)$ be the bounded derived category of coherent sheaves on S. Denote the Grothendieck group of $\mathrm{D^b}(S)$ by $K(S)$. A *Bridgeland stability condition* ([Bri07, Proposition 5.3]) $\sigma = (Z, \mathcal{A})$ on $\mathrm{D^b}(S)$ consists of a pair $(Z, \mathcal{A})$, where $Z : K(S) \to \mathbb{C}$ is a group homomorphism (called *central charge*) and $\mathcal{A} \subset \mathrm{D^b}(S)$ is the heart of a *bounded t-structure*, satisfying the following three properties (see e.g. [BM14a]).

(1) *Positivity.* For any $0 \neq E \in \mathcal{A}$ the central charge $Z(E)$ lies in the semi-closed upper half-plane $\mathbb{R}_{>0} \cdot \mathrm{e}^{(0,1]\cdot i\pi}$.

One could think of above positivity as two conditions: $\Im Z : \mathcal{A} \to \mathbb{R}_{\geq 0}$ defines a "rank" function on the abelian category $\mathcal{A}$, i.e. a non-negative function that is additive on short exact sequences; and $-\Re Z : \mathcal{A} \to \mathbb{R}$ defines a "degree" function on $\mathcal{A}$ which has the property that *"rank" zero implies "degree" positive*, i.e. $\Im Z(E) = 0 \Rightarrow -\Re Z(E) > 0$. Define the *Bridgeland slope* (might be $+\infty$ valued) as

$$\mu_\sigma(E) := \frac{-\Re(Z(E))}{\Im(Z(E))}. \tag{2.0.1}$$

For $0 \neq E \in \mathcal{A}$, we say E is *Bridgeland stable* (*semistable* respectively) if for any subobject $0 \neq F \subsetneq E$ ($0 \neq F \subseteq E$ respectively), $\mu_\sigma(F) < \mu_\sigma(E)$ ($\leq$ respectively). We also call Bridgeland (semi)stability as Z-(semi)stability.

(2) *Harder-Narasimhan property.* Every object $E \in \mathcal{A}$ has a Harder-Narasimhan filtration $0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_n = E$ such that the quotient E_i/E_{i-1}'s are Z-semistable, with $\mu_\sigma(E_1/E_0) > \mu_\sigma(E_2/E_1) > \cdots > \mu_\sigma(E_n/E_{n-1})$.

(3) *Support property.* There is a constant $C > 0$ such that, for all Z-semistable object $E \in \mathcal{A}$, we have $\|E\| \leq C|Z(E)|$, where $\|\cdot\|$ is a fixed norm on $K(X) \otimes \mathbb{R}$.

2.1 Arcara-Bertram's construction

Fix $\omega, \beta \in \mathrm{NS}(S)_{\mathbb{Q}}$ with ω ample. Arcara and Bertram [AB13] constructed a family of stability conditions $\sigma_{\omega,\beta} = (Z_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$, which generalized Bridgeland's construction on K3 or Abelian surfaces. By using the support property, the construction extended to $\omega, \beta \in N^1(S) := \mathrm{NS}(S)_{\mathbb{R}}$. The positivity of the central charge is proved by using the Bogomolov inequality and the Hodge index theorem, and the heart is a tilting of $\mathrm{Coh}(S)$ according to the imaginary part of $Z_{\omega,\beta}$.

Central charge

Denote $\mathrm{e}^L = (1, L, \frac{1}{2}L^2)$ for $L \in \mathrm{NS}(S) \otimes \mathbb{C}$. Define

$$\begin{aligned} Z_{\omega,\beta}(E) &:= -\int_S \mathrm{e}^{-(\beta+\sqrt{-1}\omega)}.\mathrm{ch}(E) \\ &= -\mathrm{ch}_2(E) + \frac{1}{2}\mathrm{ch}_0(E)\left(\omega^2 - \beta^2\right) + \mathrm{ch}_1(E).\beta \\ &\quad +\sqrt{-1}\omega.\left(\mathrm{ch}_1(E) - \mathrm{ch}_0(E)\beta\right). \end{aligned} \tag{2.1.1}$$

We often simply write $Z_{\omega,\beta}$ as Z. Using the short notation $\mathrm{ch}_i = \mathrm{ch}_i(E)$, we have

$$Z(E) = \begin{cases} -\mathrm{ch}_2 + \mathrm{ch}_1.\beta + \sqrt{-1}\omega.\mathrm{ch}_1 & \text{if } \mathrm{ch}_0 = 0; \\ \frac{1}{2\mathrm{ch}_0}\left[\left(\mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2\right) + \mathrm{ch}_0^2\omega^2 - (\mathrm{ch}_1 - \mathrm{ch}_0\beta)^2\right] & \\ +\sqrt{-1}\omega.\left(\mathrm{ch}_1 - \mathrm{ch}_0\beta\right) & \text{otherwise.} \end{cases} \tag{2.1.2}$$

Define the *phase* $\phi(E) := \frac{1}{\pi}\arg(Z(E)) \in (0,1]$ for $E \in \mathcal{A} \setminus \{0\}$. So $\mu_\sigma(E) = \tan(\pi(\phi(E) - \frac{1}{2}))$. For nonzero $E,\ F \in \mathcal{A}$, we have the equivalent relation:

$$\mu_\sigma(F) < (\leq)\mu_\sigma(E) \Longleftrightarrow \phi(F) < (\leq)\phi(E). \tag{2.1.3}$$

Torsion Pair

We refer to [HRS96] for the general torsion pair theory. The following two full subcategories $\mathcal{T}_{\omega,\eta}$ and $\mathcal{F}_{\omega,\eta}$ of $\mathrm{Coh}(S)$ are defined according to the imaginary part of $Z(E)$. For $E \in \mathrm{Coh}(S)$, denote the *Mumford slope* by

$$\mu_\omega(E) := \begin{cases} \frac{\omega \cdot \mathrm{ch}_1(E)}{\mathrm{ch}_0(E)} & \text{if } \mathrm{ch}_0(E) \neq 0; \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1.4}$$

Denote the intersection number $\eta := \omega.\beta$. If $\mathrm{ch}_0(E) \neq 0$, we can rewrite the imaginary part of Z:

$$\Im Z(E) = \mathrm{ch}_0(E) \left(\frac{\omega.\mathrm{ch}_1(E)}{\mathrm{ch}_0(E)} - \omega.\beta \right) = \mathrm{ch}_0(E) \left(\mu_\omega(E) - \eta \right).$$

For any $E \in \mathrm{Coh}(S)$, we have a unique torsion filtration ([HL10]) and the exact sequence $0 \to E_{\mathrm{tor}} \to E \to E_{\mathrm{fr}} \to 0$, where E_{tor} is the maximal subsheaf of E of dimension $\leq \dim S - 1$, and E_{fr} is torsion free. The sheaf E is called *pure* if $E_{\mathrm{tor}} = 0$. Let us denote the Harder-Narasimhan filtration of $E \in \mathrm{Coh}(S)$ with respect to the Mumford μ_ω-stability by $E_{\mathrm{tor}} = E_0 \subset E_1 \subset \cdots \subset E_{k(E)} = E$, i.e. each successive quotient $F_i := E_i/E_{i-1}$ is a torsion free μ_ω-semistable sheaf of Mumford slope $\mu_i := \mu_\omega(F_i)$ and $\mu_{\omega-\max}(E) := \mu_1 > \mu_2 > \cdots > \mu_{k(E)} =: \mu_{\omega-\min}(E)$.

Definition 2.1.1. [AB13] For an ample line bundle ω and a fixed real number η, two subcategories $\mathcal{T}_{\omega,\eta}$ and $\mathcal{F}_{\omega,\eta}$ are defined as follows:

$$\begin{aligned} \mathcal{T}_{\omega,\eta} &:= \{E \in \mathrm{Coh}(S) | \mu_{\omega-\min}(E) - \eta > 0\}, \\ \mathcal{F}_{\omega,\eta} &:= \{E \in \mathrm{Coh}(S) | \mu_{\omega-\max}(E) - \eta \leq 0\} \cup \{0\}. \end{aligned}$$

Lemma 2.1.2. *[AB13]* $(\mathcal{T}_{\omega,\eta}, \mathcal{F}_{\omega,\eta})$ *is a torsion pair of* $\mathrm{Coh}(S)$*, i.e.*

(TP1) $\mathrm{Hom}_{\mathrm{Coh}(S)}(T, F) = 0$ *for every* $T \in \mathcal{T}_{\omega,\eta}$ *and* $F \in \mathcal{F}_{\omega,\eta}$*;*

(TP2) *Every object* $E \in \mathrm{Coh}(S)$ *fits into a short exact sequence* $0 \to T \to E \to F \to 0$ *with* $T \in \mathcal{T}_{\omega,\eta}$ *and* $F \in \mathcal{F}_{\omega,\eta}$*.*

Bridgeland Stability Conditions on Surfaces

Define the heart as the tilt of the torsion pair:

$$\mathcal{A}_{\omega,\eta} := \{E \in \mathrm{D^b}(S) : \; H^p(E) = 0 \text{ for } p \neq -1, 0; \; H^{-1}(E) \in \mathcal{F}_{\omega,\eta}; \; H^0(E) \in \mathcal{T}_{\omega,\eta}\}.$$

Notation 2.1.3. We will also write $\mathcal{A}_{\omega,\eta}$ as $\mathcal{A}_{\omega,\beta}$ if we want to emphasize that

$$\eta = \omega.\beta. \tag{2.1.5}$$

Remark 2.1.4. Since $\mu_{t\omega} = t\mu_\omega$, it is clear that $\mathcal{A}_{t\omega,\beta} = \mathcal{A}_{\omega,\beta}$ for $t > 0$. By the general torsion pair theory, we obtain that $\mathcal{T}_{\omega,\eta} = \mathcal{A}_{\omega,\eta} \cap \mathrm{Coh}(S)$ and $\mathcal{F}_{\omega,\eta} = \mathcal{A}_{\omega,\eta}[-1] \cap \mathrm{Coh}(S)$.

Lemma 2.1.5. *[AB13, Corollary 2.1] Fix $\omega, \beta \in \mathrm{NS}(S)_\mathbb{R}$ with ω ample. Then $\sigma_{\omega,\beta} := (Z_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$ is a Bridgeland stability condition.*

Proof. We refer to the original proof and sketch the key steps for the positivity. Let $E \in \mathcal{A}$. If $\mathrm{ch}_0(E) = 0$, then $E \in \mathcal{T}$. So either E supports in dimension 1 and $\Im(E) = \omega.\mathrm{ch}_1(E) > 0$, or E supports in dimension 0 and $\Im(E) = 0$, $\Re(E) = -\mathrm{ch}_2(E) < 0$. Now assume that $\mathrm{ch}_0(E) \neq 0$. For $E \in \mathcal{T}$, $\Im Z(E) = \mathrm{ch}_0(E)(\mu_\omega(E) - \eta) \geq \mathrm{ch}_0(E)(\mu_{\omega-\min}(E) - \eta) > 0$. For $E = F[1]$ with $F \in \mathcal{F}$, if $\mu_{\omega-\max}(F) - \eta < 0$, we obtain $\Im Z(E) = -\Im Z(F) = -\mathrm{ch}_0(F)(\mu_\omega(F) - \eta) \geq -\mathrm{ch}_0(F)(\mu_{\omega-\max}(E) - \eta) > 0$.

We now consider the case that $E = F[1]$ with $F \in \mathcal{F}$ and $\mu_{\omega-\max}(F) - \eta = 0$. We can also assume that $\mu_\omega(F) - \eta = 0$, otherwise $\mu_\omega(F) - \eta < 0$ and then $\Im Z(F[1]) > 0$. Then $\mu_\omega(F) = \mu_{\omega-\max}(F)$ so F is μ_ω- semistable, and we have the *Bogomolov inequality* $\mathrm{ch}_1(F)^2 - 2\mathrm{ch}_0(F)\mathrm{ch}_2(F) \geq 0$. Now $\omega.(\mathrm{ch}_1(F) - \mathrm{ch}_0(F)\beta) = 0$. Since ω is ample, the *Hodge index theorem* implies that $(\mathrm{ch}_1(F) - \mathrm{ch}_0(F)\beta)^2 \leq 0$. Therefore by (2.1.2), $\Im Z(E) = -\Im Z(F) < 0$.

The general element $E \in \mathcal{A}$ is in the extension closure $\langle \mathcal{F}_{\omega,\eta}[1], \mathcal{T}_{\omega,\eta}\rangle_{\mathrm{Ext}}$ and hence satisfies the positivity. $\square$

By some physical hints (e.g. [Asp05, Section 6.2.3]), the central charge is often taken as (e.g. [Bri08, BM14a])

$$\hat{Z}_{\omega,\beta}(E) := -\int_S \mathrm{e}^{-(\beta+\sqrt{-1}\omega)}.\mathrm{ch}(E).\sqrt{\mathrm{td}(S)} = Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S} + \frac{1}{24}\mathrm{ch}_0(E)\mathrm{ch}_2(S). \tag{2.1.6}$$

Using the short notation $\mathrm{ch}_i = \mathrm{ch}_i(E)$ again, if $\mathrm{ch}_0 \neq 0$, we have

$$\begin{aligned} \hat{Z}_{\omega,\beta}(E) \;=\; & \frac{1}{2\mathrm{ch}_0}\Bigg[\left(\mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2\right) + \mathrm{ch}_0^2\left(\omega^2 + \frac{1}{8}\mathrm{K}_S^2 - \chi(\mathcal{O}_S)\right) \\ & - \left(\mathrm{ch}_1 - \mathrm{ch}_0(\beta + \frac{1}{4}\mathrm{K}_S)\right)^2\Bigg] + \sqrt{-1}\omega.\left(\mathrm{ch}_1 - \mathrm{ch}_0(\beta + \frac{1}{4}\mathrm{K}_S)\right). \end{aligned}$$

To show that $\hat{\sigma}_{\omega,\beta} := (\hat{Z}_{\omega,\beta}, \mathcal{A}_{\omega,\beta+\frac{1}{4}\mathrm{K}_S})$ is a Bridgeland stability condition, the proof of Lemma 2.1.5 breaks down unless we assume that

$$\omega^2 + \frac{1}{8}\mathrm{K}_S^2 - \chi(\mathcal{O}_S) > 0. \tag{2.1.7}$$

- Assume that (2.1.7) holds. Then $(\hat{Z}_{\omega,\beta}, \mathcal{A}_{\omega,\beta+\frac{1}{4}\mathrm{K}_S})$ is a Bridgeland stability condition.

If we use the central charge $\hat{Z}_{tH,\beta}$ for a fixed ample line bundle H, we can take t large enough, but we should be careful by taking $t \to 0$. Bridgeland originally used a weaker assumption and a different proof to show the positivity of $\hat{Z}_{tH,\beta}$ when S is a K3 surface.

Lemma 2.1.6. *[Bri08, Lemma 6.2] Let S be a K3 surface. Fix $\omega, \beta \in \mathrm{NS}(S)_{\mathbb{R}}$ with ω ample. If $\hat{Z}_{\omega,\beta}(F) \notin \mathbb{R}_{\leq 0}$ for all spherical sheaves $F \in \mathrm{Coh}(S)$ (this holds when we assume (2.1.7), i.e. $\omega^2 > 2$), then $\hat{\sigma}_{\omega,\beta} = (\hat{Z}_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$ is a Bridgeland stability condition. In particular, we can take $t \to 0$.*

In the main part of the paper, we give computations by using the stability condition $\sigma_{\omega,\beta}$ for a general smooth projective surface S. We give computations in appendix B by using $\hat{\sigma}_{\omega,\beta}$ when the surface S is K3 with extra assumption as in Lemma 2.1.6. In this case, the advantage of using $\hat{Z}_{\omega,\beta}$ instead of $Z_{\omega,\beta}$ is that $\hat{\sigma}_{\omega,\beta}$ is a *reduced* stability condition (see Remark 2.4.3).

2.2 Logarithm Todd class

For computational convenience, it is better to express the square root as

$$\sqrt{\mathrm{td}(S)} = \mathrm{e}^{\frac{1}{2}\ln \mathrm{td}(S)}$$

by introducing the *logarithm Todd class* $\ln \mathrm{td}(S)$. Let X be a smooth projective variety over $\mathbb{C}$. Let us introduce a formal variable t, and write

$$\begin{aligned} \mathrm{td}(X)(t) \;\; := \;\; & 1 + (-\frac{1}{2}\mathrm{K}_X)t + \frac{1}{12}\left(\frac{3}{2}\mathrm{K}_X^2 - \mathrm{ch}_2(X)\right)t^2 \\ & + \left(-\frac{1}{24}\mathrm{K}_X.\left(\frac{1}{2}\mathrm{K}_X^2 - \mathrm{ch}_2(X)\right)\right)t^3 + \text{ higher order of } t^4. \end{aligned}$$

Taking the logarithm with respect to t, and expressing it in the power series of t, we obtain

$$\begin{aligned}
&\ln \mathrm{td}(X)(t)\\
= \;& (-\frac{1}{2}\mathrm{K}_X)t + \frac{1}{12}\left(\frac{3}{2}\mathrm{K}_X^2 - \mathrm{ch}_2(X)\right)t^2 + \left(-\frac{1}{24}\mathrm{K}_X.\left(\frac{1}{2}\mathrm{K}_X^2 - \mathrm{ch}_2(X)\right)\right)t^3\\
& -\frac{1}{2}\left(\frac{1}{4}\mathrm{K}_X^2t^2 + 2\times(-\frac{1}{2})\mathrm{K}_X\cdot\frac{1}{12}\left(\frac{3}{2}\mathrm{K}_X^2 - \mathrm{ch}_2(X)\right)t^3\right) + \frac{1}{3}(-\frac{1}{8})\mathrm{K}_X^3t^3\\
& + \text{higher order of } t^4\\
= \;& -\frac{1}{2}\mathrm{K}_Xt - \frac{1}{12}\mathrm{ch}_2(X)t^2 + 0\cdot t^3 + \text{ higher order of } t^4.
\end{aligned}$$

Definition 2.2.1. For a smooth projective surface S or a smooth projective threefold X, we define the *logarithm Todd class* as follows:

$$\begin{aligned}
\ln \mathrm{td}(S) &:= (0, -\frac{1}{2}\mathrm{K}_S, -\frac{1}{12}\mathrm{ch}_2(S));\\
\ln \mathrm{td}(X) &:= (0, -\frac{1}{2}\mathrm{K}_X, -\frac{1}{12}\mathrm{ch}_2(X), 0).
\end{aligned}$$

For the surface S, recall the *Noether's formula* $\chi(\mathcal{O}_S) = \frac{1}{12}\left(\mathrm{K}_S^2 + \mathrm{c}_2(S)\right)$. Since $\mathrm{c}_2(S) = \frac{1}{2}\mathrm{K}_S^2 - \mathrm{ch}_2(S)$, we obtain the equivalent *Noether's formula*:

$$-\frac{1}{12}\mathrm{ch}_2(S) = \chi(\mathcal{O}_S) - \frac{1}{8}\mathrm{K}_S^2. \tag{2.2.1}$$

We will use frequently that

$$\begin{aligned}
\mathrm{e}^{\frac{1}{2}\ln \mathrm{td}(S)} &= \mathrm{e}^{-\frac{1}{4}\mathrm{K}_S}\cdot \mathrm{e}^{\frac{1}{2}\chi(\mathcal{O}_S) - \frac{1}{16}\mathrm{K}_S^2}\\
&= (1, -\frac{1}{4}\mathrm{K}_S, \frac{1}{2}\left(-\frac{1}{4}\mathrm{K}_S\right)^2)\cdot(1, 0, \frac{1}{2}\chi(\mathcal{O}_S) - \frac{1}{16}\mathrm{K}_S^2)\\
&= (1, -\frac{1}{4}\mathrm{K}_S, \frac{1}{2}\chi(\mathcal{O}_S) - \frac{1}{32}\mathrm{K}_S^2) = \sqrt{\mathrm{td}(S)}.
\end{aligned} \tag{2.2.2}$$

Example 2.2.2. We consider the following special cases.

- X is a Calabi-Yau threefold. We have $\ln \mathrm{td}(X) = (0, 0, -\frac{1}{12}\mathrm{ch}_2(X), 0) = (0, 0, \frac{1}{12}c_2(X), 0)$.
- X is an abelian threefold or a Calabi-Yau threefold of abelian type [BMS14]. Recently, Bayer, Macrì and Stellari constructed Bridgeland stability conditions on such X. In particular,

  $$\ln \mathrm{td}(X) = (0,0,0,0), \quad \text{and} \quad \hat{Z}_{\omega,\beta}(E) = Z_{\omega,\beta}(E).$$

- $X = \mathbb{P}^3$. Let H be a hyperplane class, we have $K_{\mathbb{P}^3} = -4H$ and $\mathrm{ch}_2(\mathbb{P}^3) = 2H^2$ (e.g. [LZ14, Lemma 5.1]. So $\ln \mathrm{td}(\mathbb{P}^3) = (0, 2H, -\frac{1}{6}H^2, 0)$.

2.3 The Mukai bilinear form

We refer [Huy06, Section 5.2] for the details. We still denote X a smooth projective variety over $\mathbb{C}$. Define the Mukai vector of an object $E \in \mathrm{D^b}(X)$ by

$$v(E) := \mathrm{ch}(E).\sqrt{\mathrm{td}(X)} = \mathrm{ch}(E).\mathrm{e}^{\frac{1}{2}\ln \mathrm{td}(X)} \in \oplus H^{p,p}(X) \cap H^{2p}(X,\mathbb{Q}) =: H^*_{\mathrm{alg}}(X,\mathbb{Q}).$$

Let $A(X)$ be the Chow ring of X. The Chern character gives a mapping $\mathrm{ch} : K(X) \to A(X) \otimes \mathbb{Q}$. There is a natural involution $^* : A(X) \to A(X)$,

$$\mathbf{a} = (a_0, \ldots, a_i, \ldots, a_n) \mapsto \mathbf{a}^* := (a_0, \ldots, (-1)^i a_i, \ldots, (-1)^n a_n), \tag{2.3.1}$$

Given $v \in A(X)$, we call v^* the *Mukai dual* of v. We have

$$\mathrm{ch}(E)^* = \mathrm{ch}(E^\vee), \quad \sqrt{\mathrm{td}(X)} = (\sqrt{\mathrm{td}(X)})^*.\mathrm{e}^{-\frac{1}{2}\mathrm{K}_X}. \tag{2.3.2}$$

It turns out that $v(E^\vee) = (v(E))^*.\mathrm{e}^{-\frac{1}{2}\mathrm{K}_X}$. Define the *Mukai bilinear form* (also called *Mukai pairing*) for two Mukai vector w and v by

$$\langle w,\ v\rangle_X := -\int_X w^*.v.\mathrm{e}^{-\frac{1}{2}\mathrm{K}_X}. \tag{2.3.3}$$

The Hirzebruch-Riemann-Roch theorem gives

$$\chi(F,E) = \int_X \mathrm{ch}(F^\vee).\mathrm{ch}(E).\mathrm{td}(X) = -\langle v(F),\ v(E)\rangle_X. \tag{2.3.4}$$

Now let us consider $X = S$ a smooth projective surface. Denoting the Mukai vectors by its component, and using equation (2.2.2), we have

$$\begin{aligned} v(E) &= (v_0(E), v_1(E), v_2(E)) = \mathrm{ch}(E).\mathrm{e}^{\frac{1}{2}\ln \mathrm{td}(S)} \\ &= (\mathrm{ch}_0, \mathrm{ch}_1 - \frac{1}{4}\mathrm{ch}_0\mathrm{K}_S, \mathrm{ch}_2 - \frac{1}{4}\mathrm{ch}_1.\mathrm{K}_S + \frac{1}{2}\mathrm{ch}_0\left(\chi(\mathcal{O}_S) - \frac{1}{16}\mathrm{K}_S^2\right)). \end{aligned} \tag{2.3.5}$$

By (2.3.3) the Mukai paring of $w = (w_0, w_1, w_2)$ and $v = (v_0, v_1, v_2)$ is

$$\begin{aligned} \langle w,\ v\rangle_S &= -\int_S (w_0, -w_1, w_2).(v_0, v_1, v_2).(1, -\frac{1}{2}\mathrm{K}_S, \frac{1}{8}\mathrm{K}_S^2) \\ &= w_1.v_1 - w_0(v_2 - \frac{1}{2}v_1.\mathrm{K}_S) - v_0(w_2 + \frac{1}{2}w_1.\mathrm{K}_S) - \frac{1}{8}w_0v_0\mathrm{K}_S^2. \end{aligned} \tag{2.3.6}$$

A direct computation shows that Mukai pairing in the surface case is symmetric if and only if the K_S is trivial.

2.4 Central charge in terms of the Mukai bilinear form

Is is well know that the central charge can be represented by a Mukai bilinear form (see Remark 2.4.2). Our key observation is to express the correspondence complex Mukai vector explicitly. The further decomposition formula is based on this expression.

Lemma 2.4.1. *The central charges in terms of Mukai bilinear form are given as follows:*

$$\begin{aligned} Z_{\omega,\beta}(E) &= \langle \mho_Z, v(E)\rangle_S, \quad \textit{where} \quad \mho_{Z_{\omega,\beta}} := \mathrm{e}^{\beta-\frac{3}{4}\mathrm{K}_S+\sqrt{-1}\omega+\frac{1}{24}\mathrm{ch}_2(S)}; && (2.4.1)\\ \hat{Z}_{\omega,\beta}(E) &= \langle \mho_{\hat{Z}}, v(E)\rangle_S, \quad \textit{where} \quad \mho_{\hat{Z}_{\omega,\beta}} := \mathrm{e}^{\beta-\frac{1}{2}\mathrm{K}_S+\sqrt{-1}\omega}. && (2.4.2)\end{aligned}$$

Here the Mukai vector $\mho_{Z_{\omega,\beta}}$ (or simply $\mho_Z$) means that

$$\begin{aligned} \mho_{Z_{\omega,\beta}} &= \mathrm{e}^{\beta-\frac{3}{4}\mathrm{K}_S+\sqrt{-1}\omega}\cdot \mathrm{e}^{\frac{1}{24}\mathrm{ch}_2(S)}\\ &= \left(1, \beta-\frac{3}{4}\mathrm{K}_S+\sqrt{-1}\omega, \frac{1}{2}(\beta-\frac{3}{4}\mathrm{K}_S+\sqrt{-1}\omega)^2\right)\cdot\left(1,0,\frac{1}{24}\mathrm{ch}_2(S)\right)\\ &= \left(1, \beta-\frac{3}{4}\mathrm{K}_S, -\frac{1}{2}\omega^2+\frac{1}{2}(\beta-\frac{3}{4}\mathrm{K}_S)^2-\frac{1}{2}\left[\chi(\mathcal{O}_S)-\frac{1}{8}\mathrm{K}_S^2\right]\right)\\ &\quad+\sqrt{-1}\left(0,\omega,(\beta-\frac{3}{4}\mathrm{K}_S).\omega\right). && (2.4.3)\end{aligned}$$

Proof.

$$\begin{aligned} Z_{\omega,\beta}(E) &= -\int_S \mathrm{e}^{-(\beta+\sqrt{-1}\omega)}.\mathrm{ch}(E)\\ &= -\int_S \mathrm{e}^{-\left(\beta+\ln\mathrm{td}(S)+\sqrt{-1}\omega\right)}.\sqrt{\mathrm{td}(S)}.\mathrm{ch}(E).\sqrt{\mathrm{td}(S)}.\end{aligned}$$

Denote $\mathrm{ch}(F^\vee) := \mathrm{e}^{-\left(\beta+\ln\mathrm{td}(S)+\sqrt{-1}\omega\right)}$. Then

$$\begin{aligned} \mathrm{ch}(F)^* &= \mathrm{ch}(F^\vee) = \mathrm{e}^{-\left(\beta-\frac{1}{2}\mathrm{K}_S+\sqrt{-1}\omega\right)+\frac{1}{12}\mathrm{ch}_2(S)}\\ &= \left(1, -(\beta-\frac{1}{2}\mathrm{K}_S+\sqrt{-1}\omega), \frac{1}{2}(\beta-\frac{1}{2}\mathrm{K}_S+\sqrt{-1}\omega)^2+\frac{1}{12}\mathrm{ch}_2(S)\right).\end{aligned}$$

So

$$\begin{aligned} \mathrm{ch}(F) &= \left(1, (\beta-\frac{1}{2}\mathrm{K}_S+\sqrt{-1}\omega), \frac{1}{2}(\beta-\frac{1}{2}\mathrm{K}_S+\sqrt{-1}\omega)^2+\frac{1}{12}\mathrm{ch}_2(S)\right)\\ &= \mathrm{e}^{\left(\beta-\frac{1}{2}\mathrm{K}_S+\sqrt{-1}\omega\right)+\frac{1}{12}\mathrm{ch}_2(S)}.\end{aligned}$$

Therefore,

$$Z_{\omega,\beta}(E) = -\int_S \mathrm{ch}(F^\vee).\sqrt{\mathrm{td}(S)}.\mathrm{ch}(E).\sqrt{\mathrm{td}(S)} = \langle v(F), v(E)\rangle_S. \qquad (2.4.4)$$

So

$$\mho_{Z_{\omega,\beta}} = v(F) = \mathrm{ch}(F).\mathrm{e}^{\frac{1}{2}\ln \mathrm{td}(S)} = \mathrm{e}^{\beta - \frac{3}{4}\mathrm{K}_S + \sqrt{-1}\omega + \frac{1}{24}\mathrm{ch}_2(S)}.$$

By using the Noether's formula (2.2.1) and direct computation, we get the concrete expression of $\mho_Z$. Similarly, we obtain $\mho_{\hat{Z}_{\omega,\beta}} = \mathrm{e}^{\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega}$. $\square$

Remark 2.4.2. Recall, in [Bri08], a Bridgeland stability condition $\sigma = (Z, \mathcal{A})$ on $\mathrm{D^b}(S)$ is said to be *numerical* if the central charge Z takes the form $Z(E) = \langle \pi(\sigma), v(E)\rangle_S$ for some vector $\pi(\sigma) \in K_{\mathrm{num}}(S) \otimes \mathbb{C}$. As in [Huy14, Remark 4.33], we further *assume* the numerical Bridgeland stability factors through $K_{\mathrm{num}}(S)_{\mathbb{Q}} \otimes \mathbb{C} \to H^*_{\mathrm{alg}}(S, \mathbb{Q}) \otimes \mathbb{C}$. Therefore $\pi(\sigma) \in H^*_{\mathrm{alg}}(S, \mathbb{Q}) \otimes \mathbb{C}$. Let $\mathrm{Stab}(S)$ be the *stability manifold*, which is a $\mathbb{C}$-manifold of dimension $K_{\mathrm{num}}(S) \otimes \mathbb{C}$. Denote $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ the universal cover of the group of orientation-preserving automorphism of $\mathbb{R}^2$. Denote $\mathrm{Aut}(\mathrm{D^b}(S))$ the group of triangulated, $\mathbb{C}$-linear auto-equivalences of $\mathrm{D^b}(S)$. These two groups act on $\mathrm{Stab}(S)$. One is a right action of the group $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$. The other is a left action of the group $\mathrm{Aut}(\mathrm{D^b}(S))$ on $\mathrm{Stab}(S)$. The two actions commute [Bri07, Lemma 8.2]. A Bridgeland stability condition σ on $\mathrm{D^b}(S)$ is said to be *geometric* if all skyscraper sheaves $\mathcal{O}_x$, $x \in S$, are σ-stable of the same phase. We can set the phase to be 1 after a group action. For numerical geometric stability conditions with skyscraper sheaves of phase 1 on surfaces, the heart $\mathcal{A}$ must be of the form $\mathcal{A}_{\omega,\beta}$ by the classification theorem of Bridgeland [Bri07, Proposition 10.3] and Huybrechts [Huy14, Theorem 4.39]. Denote by $U(S) \subset \mathrm{Stab}(S)$ the open subset consisting of geometric stability conditions. Let $\mathrm{Stab}^\dagger(S) \subset \mathrm{Stab}(S)$ be the connected component containing $U(S)$. Now we have $\pi(\sigma_{\omega,\beta}) = \mho_Z \in H^*_{\mathrm{alg}}(S, \mathbb{Q}) \otimes \mathbb{C}$, and $\sigma_{\omega,\beta} \in U(S)$.

Remark 2.4.3. Recall in [Bri09, BB13] that a numerical stability condition σ is called *reduced* if the corresponding $\pi(\sigma)$ satisfies $\langle \pi(\sigma), \pi(\sigma)\rangle_S = 0$. Denote

$$\mathrm{Stab}^\dagger_{\mathrm{red}} := \{\sigma \in \mathrm{Stab}^\dagger |\, \langle \pi(\sigma), \pi(\sigma)\rangle_S = 0\}.$$

This notion is the *Hodge theoretic restrictions* on the central charge. By the basic computation, we obtain that $\langle \mho_Z, \mho_Z\rangle_S = \chi(\mathcal{O}_S) - \frac{1}{4}\mathrm{K}_S^2$ and $\langle \mho_{\hat{Z}}, \mho_{\hat{Z}}\rangle_S = -\frac{1}{8}\mathrm{K}_S^2$. So $\sigma_{\omega,\beta}$ is not reduced in general. But $\hat{\sigma}_{\omega,\beta}$ is reduced when the surface is K3 with assumptions as in Lemma 2.1.6. In this case, there is a quotient stack $\mathcal{L}_{\mathrm{Kah}} := \mathrm{Stab}^{\dagger}_{\mathrm{red}}(S)/\mathrm{Aut}_{\mathrm{CY}}(S)$ ([BB13, Section 7]). There is a furthermore quotient

$$\mathcal{M}_{\mathrm{Kah}}(S) = \mathcal{L}_{\mathrm{Kah}}/\mathbb{C}^*,$$

which is viewed as a mathematical version of the *stringy Kähler moduli space* of the K3 surface S. Here $\mathrm{Aut}_{\mathrm{CY}}(\mathrm{S})$ consists of some special autoequivalence $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S))$ which induces Hodge isometry on $H^*(S)$.

2.5 Maciocia's theorem on the structure of the walls

Let $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ be of Bogomolov type, i.e. $\mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2 \geq 0$. Denote ch' the potential destabilized Chern characters. A *potential wall* is defined as $W(\mathrm{ch}, \mathrm{ch}') := \{\sigma \in \mathrm{Stab}(S) | \, \mu_\sigma(\mathrm{ch}) = \mu_\sigma(\mathrm{ch}')\}$. A potential wall $W(\mathrm{ch}, \mathrm{ch}')$ is a *Bridgeland wall* if there is a $\sigma \in \mathrm{Stab}(S)$ and objects E, $F \in \mathcal{A}_\sigma$ such that $\mathrm{ch}(E) = \mathrm{ch}$, $\mathrm{ch}(F) = \mathrm{ch}'$ and $\mu_\sigma(E) = \mu_\sigma(F)$. There is a wall-chamber structure on $\mathrm{Stab}(S)$ [Bri07, Bri08, Tod08]. Bridgeland walls are codimension 1 in $\mathrm{Stab}(S)$, which separate $\mathrm{Stab}(S)$ into chambers. Let E be an object that is σ_0-stable for a stability condition σ_0 in some chamber $\mathtt{C}$. Then E is σ-stable for any $\sigma \in \mathtt{C}$.

We follow notations in [Mac14, Section 2] (but use H instead of ω therein). Fix an ample divisor H and another divisor $\gamma \in H^\perp$. Choose

$$\begin{cases} \omega := tH, \\ \beta := sH + u\gamma, \end{cases} \tag{2.5.1}$$

for some real numbers t, s, u, with t positive. There is a half 3-space of stability conditions

$$\Omega_{\omega,\beta} = \Omega_{tH,sH+u\gamma} := \{\sigma_{tH,sH+u\gamma} \mid t > 0\} \subset \mathrm{Stab}^{\dagger}(S), \tag{2.5.2}$$

which should be considered as the u-indexed family of half planes

$$\Pi_{(H,\gamma,u)} := \{\sigma_{tH,sH+u\gamma} \mid t > 0, \ u \text{ is fixed.}\}.$$

Definition 2.5.1. A *Maciocia's coordinate* with respect to the triple (H, γ, u), is a choice of an ample divisor H, another divisor $\gamma \in H^\perp$, and a real number u, such that the stability conditions $\sigma_{\omega,\beta}$ are on the half plane $\Pi_{(H,\gamma,u)}$ $(t > 0)$ with coordinates (s, t) as equation (2.5.1). We simply call this as fixing a Maciocia's coordinate (H, γ, u).

Fix the Chern characters $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2) := (x, \mathrm{ch}_1, z)$ and assume that ch are of Bogomolov type, i.e. $\mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2 \geq 0$. Decompose $\mathrm{ch}_1 = y_1 H + y_2\gamma + \delta$, where y_1, y_2 are real coefficients, and $\delta \in \{H, \gamma\}^{\perp}$, i.e. $\delta.H = 0$ and $\delta.\gamma = 0$. Write the potential destabilizing Chern characters as $\mathrm{ch}' := (\mathrm{ch}_0', \mathrm{ch}_1', \mathrm{ch}_2') = (r, c_1 H + c_2\gamma + \delta', \chi)$, where c_1, c_2 are real coefficients, and $\delta' \in \{H, \gamma\}^{\perp}$. Denote $g := H^2$, $-d := \gamma^2$. It is known by *Hodge index theorem* that $d \geq 0$, and $d = 0$ if and only if $\gamma = 0$. We summarize above relations in (1.5.1, 1.5.2, 1.5.3, 1.5.4).

Theorem 2.5.2. *[Mac14, Section 2] Fix a Maciocia's coordinate (H, γ, u). The potential walls $W(\mathrm{ch}, \mathrm{ch}')$ (for the fixed ch and different potential destabilizing Chern characters ch') in the (s, t)-half-plane $\Pi_{(H,\gamma,u)}$ $(t > 0)$ are given by nested semicircles with center $(C, 0)$ and radius $R = \sqrt{D + C^2}$:*

$$(s - C)^2 + t^2 = D + C^2, \tag{2.5.3}$$

where $C = C(\mathrm{ch}, \mathrm{ch}')$ and $D = D(\mathrm{ch}, \mathrm{ch}')$ are given by

$$C(\mathrm{ch}, \mathrm{ch}') = \frac{x\chi - rz + ud(xc_2 - ry_2)}{g(xc_1 - ry_1)}, \tag{2.5.4}$$

$$D(\mathrm{ch}, \mathrm{ch}') = \frac{2zc_1 - 2c_2udy_1 - xu^2dc_1 + 2y_2udc_1 - 2\chi y_1 + ru^2dy_1}{g(xc_1 - ry_1)}. \tag{2.5.5}$$

- *If $\mathrm{ch}_0 = x \neq 0$, then*

$$D = \frac{ud(2y_2 - ux) + 2z}{gx} - \frac{2y_1}{x}C. \tag{2.5.6}$$

- *If $\mathrm{ch}_0 = 0$ and $\mathrm{ch}_1.H > 0$, i.e. $x = 0$ and $y_1 > 0$, then the center $(C, 0)$ is independent of ch', and $C = \frac{z+duy_2}{gy_1}$.*

By nested we mean that if W_1 and W_2 are two distinct (semicircular) potential walls in $\Pi_{(H,\gamma,u)}$ then either W_1 is entirely contained in the interior of the semicircle W_2 or vice-versa. The following lemma is an easy exercise.

Lemma 2.5.3. *Denote $W(\mathrm{ch}, \mathrm{ch}')$ the potential wall for ch for the potential destabilizing Chern characters ch'. Let $\sigma \in W(\mathrm{ch}, \mathrm{ch}')$. Then we have*
$\mu_\sigma(\mathrm{ch}) = \mu_\sigma(k\mathrm{ch}') = \mu_\sigma(\mathrm{ch} - \mathrm{ch}')$ for $k \in \mathbb{Z}\backslash 0$; $W(\mathrm{ch}, \mathrm{ch}') = W(\mathrm{ch}, \mathrm{ch} - \mathrm{ch}')$;

$C(\mathrm{ch}, \mathrm{ch}') = C(\mathrm{ch}, \mathrm{ch} - \mathrm{ch}')$; $D(\mathrm{ch}, \mathrm{ch}') = D(\mathrm{ch}, \mathrm{ch} - \mathrm{ch}')$.
If $\mathrm{ch}_0' = r \neq 0$, *we have*

$$D = \frac{ud(2c_2 - ur) + 2\chi}{gr} - \frac{2c_1}{r}C. \tag{2.5.7}$$

It may happen that for Chern characters $\mathrm{ch}', \mathrm{ch}''$, where ch'' is linearly independent of ch' and $\mathrm{ch} - \mathrm{ch}'$, the walls coincide $W(\mathrm{ch}, \mathrm{ch}') = W(\mathrm{ch}, \mathrm{ch}'')$. Then at the wall, the contracted loci have several connected components, for example [BM14b, Example 14.4].

2.6 Duality induced by derived dual

Lemma 2.6.1. *[Mar13, Theorem 3.1] The functor* $\Phi(\cdot) := R\mathcal{H}om(\cdot, \mathcal{O}_S)[1]$ *induces an isomorphism between the Bridgeland moduli spaces* $M_{\omega,\beta}(\mathrm{ch})$ *and* $M_{\omega,-\beta}(-\mathrm{ch}^*)$ *provided these moduli spaces exist and* $Z_{\omega,\beta}(\mathrm{ch})$ *belongs to the open upper half plane.*

Proof. This is a variation of Martinez's duality theorem [Mar13, Theorem 3.1], where the duality functor is taken as $R\mathcal{H}om(\cdot, \omega_S)[1]$. □

Corollary 2.6.2. *Fix the Chern characters* $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. *Assume that* $Z_{\omega,\beta}(\mathrm{ch})$ *belongs to the open upper half plane. The wall-chamber structures of* $\sigma_{\omega,\beta}$ *for* ch *is dual to the wall-chamber structures of* $\Phi(\sigma_{\omega,\beta})$ *for* $\Phi(\mathrm{ch}) := -\mathrm{ch}^* = (-\mathrm{ch}_0, \mathrm{ch}_1, -\mathrm{ch}_2)$ *in the sense that*

$$\Phi(\sigma_{\omega,\beta}) = \sigma_{\omega,-\beta}. \tag{2.6.1}$$

Applying Φ *again, we have* $\Phi \circ \Phi(\sigma_{\omega,\beta}) = \sigma_{\omega,\beta}$. *Moreover, if we fix a Maciocia's coordinate* (H, γ, u), *then* $\sigma_{\omega,\beta} \in \Pi_{(H,\gamma,u)}$ *with coordinates* (s, t) *is dual to* $\Phi(\sigma_{\omega,\beta}) \in \Pi_{(H,\gamma,-u)}$ *with coordinates* $(-s, t)$.

- *If* $\sigma_{\omega,\beta} \in \mathtt{C}$, *where* $\mathtt{C}$ *is a chamber for* ch *in* $\Pi_{(H,\gamma,u)}$, *then we have* $\Phi(\sigma_{\omega,\beta}) \in \mathtt{DC}$, *where* $\mathtt{DC}$ *is the corresponding chamber for* $\Phi(\mathrm{ch})$ *in* $\Pi_{(H,\gamma,-u)}$.
- *If* $\sigma := \sigma_{\omega,\beta} \in W(\mathrm{ch}, \mathrm{ch}')$ *in* $\Pi_{(H,\gamma,u)}$, *then* $\Phi(\sigma) \in W(-\mathrm{ch}^*, -\mathrm{ch}'^*)$ *in* $\Pi_{(H,\gamma,-u)}$, *and there are relations*
  $\mu_{\Phi(\sigma)}(-\mathrm{ch}^*) = -\mu_\sigma(\mathrm{ch})$; $C_{\Phi(\sigma)}(-\mathrm{ch}^*, -\mathrm{ch}'^*) = -C_\sigma(\mathrm{ch}, \mathrm{ch}')$;
  $D_{\Phi(\sigma)}(-\mathrm{ch}^*, -\mathrm{ch}'^*) = D_\sigma(\mathrm{ch}, \mathrm{ch}')$; $R_{\Phi(\sigma)}(-\mathrm{ch}^*, -\mathrm{ch}'^*) = R_\sigma(\mathrm{ch}, \mathrm{ch}')$.

Proof. The proof is a direct computation. $\square$

Remark 2.6.3. The assumption that $Z_{\omega,\beta}(\mathrm{ch})$ belongs to the open upper half plane means exactly that we exclude the case $\Im Z_{\omega,\beta}(\mathrm{ch}) = 0$, which is equivalent to the following three subcases:

- $\mathrm{ch} = (0, 0, n)$ for some positive integer n; or
- $\mathrm{ch}_0 > 0$ and $\Im Z_{\omega,\beta}(\mathrm{ch}) = 0$; or
- $\mathrm{ch}_0 < 0$ and $\Im Z_{\omega,\beta}(\mathrm{ch}) = 0$.

We call the first subcase as the trivial chamber, the second subcase as the Uhlenbeck wall and the third subcase as the dual Uhlenbeck wall. The details are given in Definition 4.3.1.

Chapter 3

Bayer-Macrì's nef line bundle theory

In this Chapter, we review Bayer-Macrì's nef line bundle theory [BM14a] over Bridgeland moduli spaces.

3.1 Construction of Bayer-Macrì line bundle

Let S be a smooth projective surface over $\mathbb{C}$. Let $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(S)$ be a stability condition, and $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ be a choice of Chern characters. Assume that we are given a *flat family* [BM14a, Definition 3.1] $\mathcal{E} \in \mathrm{D^b}(M \times S)$ of σ-semistable objects of class ch parametrized by a proper algebraic space M of finite type over $\mathbb{C}$. Recall the equation (C.1.1) that $N^1(M) = \mathrm{NS}(M)_{\mathbb{R}}$ is the group of real Cartier divisors modulo numerical equivalence; dually write $N_1(M)$ as the group of real 1-cycles modulo numerical equivalence with respect to the intersection paring with Cartier divisors. Bayer-Macrì's numerical Cartier divisor class $\ell_{\sigma,\mathcal{E}} \in N^1(M) = \mathrm{Hom}(N_1(M), \mathbb{R})$ is defined as follows: for any projective integral curves $C \subset M$,

$$\ell_{\sigma,\mathcal{E}}([C]) = \ell_{\sigma,\mathcal{E}}.C := \Im\left(-\frac{Z\big(\Phi_{\mathcal{E}}(\mathcal{O}_C)\big)}{Z(\mathrm{ch})}\right) = \Im\left(-\frac{Z\big((p_S)_*\mathcal{E}|_{C\times S}\big)}{Z(\mathrm{ch})}\right), \quad (3.1.1)$$

where $\Phi_{\mathcal{E}} \colon \mathrm{D^b}(M) \to \mathrm{D^b}(S)$ is the Fourier-Mukai functor with kernel $\mathcal{E}$, and $\mathcal{O}_C$ is the structure sheaf of C.

Theorem 3.1.1. *[BM14a, Theorem 1.1] The divisor class $\ell_{\sigma,\mathcal{E}}$ is nef on M. In addition, we have $\ell_{\sigma,\mathcal{E}}.C = 0$ if and only if for two general points $c, c' \in C$, the corresponding objects $\mathcal{E}_c, \mathcal{E}_{c'}$ are S-equivalent.*

Here two semistable objects are *S-equivalent* if their Jordan-Hölder filtrations into stable factors of the same phase have identical stable factors.

Definition 3.1.2. Let $\mathtt{C}$ be a Bridgeland chamber for ch. Assume the existence of the moduli space $M_\sigma(\mathrm{ch})$ for $\sigma \in \mathtt{C}$ and a universal family $\mathcal{E}$. Then $M_{\mathtt{C}}(\mathrm{ch}) := M_\sigma(\mathrm{ch})$ is constant for $\sigma \in \mathtt{C}$. Theorem 3.1.1 yields a map,

$$\begin{aligned} \ell : \overline{\mathtt{C}} \ &\longrightarrow \ \mathrm{Nef}(M_{\mathtt{C}}(\mathrm{ch})) \\ \sigma \ &\mapsto \ \ell_{\sigma,\mathcal{E}} \end{aligned}$$

which is called the *local Bayer-Macrì map* for the chamber $\mathtt{C}$ for ch.

For numerical geometric Bridgeland stability conditions as in Remark 2.4.2, a decomposition of the local Bayer-Macrì map is given in [BM14a]. For any $\sigma \in \mathrm{Stab}^\dagger(S)$, we can assume that $\sigma = \sigma_{\omega,\beta}$ after a group action, i.e. skyscraper sheaves are stable of phase 1. Recall that $\sigma_{\omega,\beta} = (Z_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$, and $Z_{\omega,\beta}(E) = \langle \mho_Z, v(E)\rangle_S$ as in (2.4.1). For the fixed Chern characters ch, denote $\mathbf{v} := v(\mathrm{ch}) = \mathrm{ch}\cdot \mathrm{e}^{\frac{1}{2}\ln \mathrm{td}(S)}$ the corresponding Mukai vector. The local Bayer-Macrì map is the composition of the following three maps:

$$\mathrm{Stab}^\dagger(S) \xrightarrow{\mathcal{Z}} H^*_{\mathrm{alg}}(S,\mathbb{Q})\otimes\mathbb{C} \xrightarrow{\mathcal{I}} \mathbf{v}^\perp \xrightarrow{\theta_{\mathtt{C},\mathcal{E}}} N^1(M_{\mathtt{C}}(\mathrm{ch})).$$

- The first map $\mathcal{Z}$ forgets the heart: $\mathcal{Z}(\sigma_{\omega,\beta}) := \mho_Z$. This map is the dual version (with respect to Mukai paring) of the original Bridgeland map [Bri08, Section 8].
- The second map $\mathcal{I}$ forgets more: for any $\mho \in H^*_{\mathrm{alg}}(S,\mathbb{Q})\otimes\mathbb{C}$, define $\mathcal{I}(\mho) := \Im\frac{\mho}{-\langle\mho,\mathbf{v}\rangle_S}$. One can check $\mathcal{I}(\mho) \in \mathbf{v}^\perp$ (this also follows from the Lemma 3.2.2), where the perpendicular relation is with respect to Mukai paring:
$$\mathbf{v}^\perp := \{w \in H^*_{\mathrm{alg}}(S,\mathbb{Q})\otimes\mathbb{R} \mid \langle w,\mathbf{v}\rangle_S = 0\}. \tag{3.1.2}$$
- The third map $\theta_{\mathtt{C},\mathcal{E}}$ is the *algebraic Mukai morphism* (not only K3). More precisely, for a fixed Mukai vector $w \in \mathbf{v}^\perp$, and an integral curve $C \subset M_{\mathtt{C}}(\mathrm{ch})$,
$$\theta_{\mathtt{C},\mathcal{E}}(w).[C] := \langle w, v(\Phi_{\mathcal{E}}(\mathcal{O}_C))\rangle_S. \tag{3.1.3}$$

When the surface S is K3, and $\mathbf{v}$ is positive primitive with $\langle \mathbf{v}, \mathbf{v} \rangle_S > 0$, a well-known theorem of Yoshioka ([Yos02], see also [BM14a, Theorem 5.6]) says that the Mukai morphism is an isomorphism. Meachan and Zhang [MZ14] recently studied the Mukai vector $\mathbf{v}$ of O'Grady type on K3 surfaces.

Donaldson morphism

The Mukai morphism $\theta_{\mathsf{C},\mathcal{E}}$ is the dual version of the *Donaldson morphism* $\lambda_{\mathcal{E}}$ (Lemma 3.1.4) which was introduced by Le Potier [LeP92] (see also [HL10, Definition 8.1.1] and [BM14a, Section 4]). Let $M = M_{\mathsf{C}}(\mathrm{ch})$. Denote $\mathrm{D}_{M\text{-perf}}(M \times S)$ for the category of *M-perfect complexes.* (An M-perfect complex is a complex of $\mathcal{O}_{M\times S}$-modules which locally, over M, is quasi-isomorphic to a bounded complex of coherent shaves which are flat over M.) The Euler characteristic gives a well-defined pairing

$$\chi : K(\mathrm{D^b}(\mathrm{Coh}(M))) \times K(\mathrm{D_{perf}}(\mathrm{Coh}(M))) \to \mathbb{Z}$$

between the Grothendieck K-groups of the bounded derived categories of coherent sheaves $\mathrm{D^b}(\mathrm{Coh}(M))$ and of perfect complexes $\mathrm{D_{perf}}(\mathrm{Coh}(M))$. Taking the quotient with respect to the kernel of χ on each side we obtain the induced perfect pairing

$$\chi : K_{\mathrm{num}}(M) \times K^{\mathrm{perf}}_{\mathrm{num}}(M) \to \mathbb{Z}$$

between the numerical Grothendieck K-groups of the bounded derived categories of coherent sheaves $K_{\mathrm{num}}(M)$ and of perfect complexes $K^{\mathrm{perf}}_{\mathrm{num}}(M)$. It follows from equations (2.3.2, 2.3.4) that

$$\chi(a \cdot b) := \chi(a^{\vee}, b) = \int_S \mathrm{ch}(a).\mathrm{ch}(b).\mathrm{td}(S) = -\langle v(\mathrm{ch}(a)^*), v(\mathrm{ch}(b)) \rangle_S, \quad (3.1.4)$$

where $a \in K_{\mathrm{num}}(M)$ and $b \in K^{\mathrm{perf}}_{\mathrm{num}}(M)$.

Definition 3.1.3. [BM14a, Definition 4.3, Definition 5.4] Fix $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. Let $M = M_{\mathsf{C}}(\mathrm{ch})$. Define a group homomorphism

$$\lambda_{\mathcal{E}} : \mathrm{ch}^{\sharp} \to N^1(M)$$

as the composition (called the *Donaldson morphism*)

$$\mathrm{ch}^{\sharp} \xrightarrow{p_S^*} K^{\mathrm{perf}}_{\mathrm{num}}(M \times S)_{\mathbb{R}} \xrightarrow{\cdot[\mathcal{E}]} K^{\mathrm{perf}}_{\mathrm{num}}(M \times S)_{\mathbb{R}} \xrightarrow{(p_M)_*} K^{\mathrm{perf}}_{\mathrm{num}}(M)_{\mathbb{R}} \xrightarrow{\det} N^1(M),$$

where

$$\mathrm{ch}^{\sharp} := \{a \in K_{\mathrm{num}}(S)_{\mathbb{R}} : \chi(a \cdot \mathrm{ch}) = 0\}.$$

Lemma 3.1.4. *[BM14a, Proposition 4.4, Remark 5.5] Fix* ch *and write its Mukai vector by* $\mathbf{v} = v(\mathrm{ch})$*. The Mukai morphism is the dual version of Donaldson morphism in the sense that*

$$\theta_{\mathsf{C},\mathcal{E}}(v(\mathrm{ch}(a))) = -\lambda_{\mathcal{E}}(\mathrm{ch}(a)^{*}), \tag{3.1.5}$$

where $v(\mathrm{ch}(a)) := \mathrm{ch}(a).\mathrm{e}^{\frac{1}{2}\ln \mathrm{td}(S)} \in v(\mathrm{ch})^{\perp} = \mathbf{v}^{\perp}$ *and* $a^{\vee} \in \mathrm{ch}^{\sharp}$*.*

The surjectivity of the Mukai morphism (or the Donaldson morphism) is not known in general. We compute the image of the local Bayer-Macrì map in Theorem 5.4.4.

The assumption on the family $\mathcal{E}$ can be relax to the case of *quasi-family* [BM14a, Definition 4.5].

3.2 Computation by using $\mho_Z$

Definition 3.2.1. Define $w_{\sigma_{\omega,\beta}}(\mathrm{ch}) := -\Im\left(\overline{\langle \mho_Z, \mathbf{v}\rangle_S} \cdot \mho_Z\right)$, where $\mathbf{v}$ is the Mukai vector $v(\mathrm{ch})$. We simply write it as $w_{\omega,\beta}$ or w_{σ}.

By the definition, $w_\sigma = |Z(E)|^2 \mathcal{I}(\mho_Z)$. So $w_\sigma \xlongequal{\mathbb{R}_+} \mathcal{I}(\mho_Z)$ up to such positive scalar, and $w_\sigma \in \mathbf{v}^{\perp}$ by the following lemma. The Mukai vector $w_{\omega,\beta}(\mathrm{ch})$ is crucial for computing the local Bayer-Macrì map. We give a preliminary computation lemma.

Lemma 3.2.2. *Fix the Chern characters* $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$*. The line bundle class* $\ell_{\sigma_{\omega,\beta}} \in N^1(M_{\sigma_{\omega,\beta}}(\mathrm{ch}))$ *(if exists) is given by*

$$\ell_{\sigma_{\omega,\beta}} \xlongequal{\mathbb{R}_+} \theta_{\sigma,\mathcal{E}}(w_{\omega,\beta}), \tag{3.2.1}$$

where $w_{\omega,\beta} \in \mathbf{v}^{\perp}$ *is given by*

$$w_{\omega,\beta} = (\Im Z(\mathrm{ch}))\, \Re\mho_Z - (\Re Z(\mathrm{ch}))\, \Im\mho_Z. \tag{3.2.2}$$

- *If* $\Im Z(\mathrm{ch}) = 0$*, then* $w_\sigma \xlongequal{\mathbb{R}_+} \Im\mho_Z$*.*

- *If* $\Im Z(\mathrm{ch}) > 0$, *then*

$$\begin{aligned} w_\sigma &\overset{\mathbb{R}_+}{=\!=} \mu_\sigma(\mathrm{ch})\Im\mho_Z + \Re\mho_Z \\ &= \left(0, \mu_\sigma(\mathrm{ch})\omega + \beta, -\frac{3}{4}\mathrm{K}_S.[\mu_\sigma(\mathrm{ch})\omega + \beta]\right) \\ &\quad + \left(1, -\frac{3}{4}\mathrm{K}_S, -\frac{1}{2}\chi(\mathcal{O}_S) + \frac{11}{32}\mathrm{K}_S^2)\right) \\ &\quad + \left(0, 0, \beta.[\mu_\sigma(\mathrm{ch})\omega + \beta] - \frac{1}{2}(\omega^2 + \beta^2)\right). \end{aligned} \tag{3.2.3}$$

Proof. The equation (3.2.1) is nothing but the definition of $w_{\omega,\beta}$. Taking the complex conjugate of equation (2.4.1) we get $\overline{\langle \mho_Z, \mathbf{v}\rangle_S} = \Re Z(\mathrm{ch}) - \sqrt{-1}\Im Z(\mathrm{ch})$. The relation (3.2.2) thus comes from the definition of w_σ. By the definition of $\mho_Z$, we have $\langle \Re\mho_Z, \mathbf{v}\rangle_S + \sqrt{-1}\langle \Im\mho_Z, \mathbf{v}\rangle_S = \langle \mho_Z, \mathbf{v}\rangle_S = \Re Z(\mathrm{ch}) + \sqrt{-1}\Im Z(\mathrm{ch})$. Then $\langle w_\sigma, \mathbf{v}\rangle_S = (\Im Z(\mathrm{ch}))\langle \Re\mho_Z, \mathbf{v}\rangle_S - (\Re Z(\mathrm{ch}))\langle \Im\mho_Z, \mathbf{v}\rangle_S = 0$. If $\Im Z(\mathrm{ch}) > 0$, we divide equation (3.2.2) by this positive number and obtain equation (3.2.3). We then use equation (2.4.3) to get the concrete formula. □

We *cannot* apply $\theta_{\sigma,\mathcal{E}}$ to those three terms in the right side of equation (3.2.3) separately since in general none of them is in $\mathbf{v}^\perp$. The refined decompositions are given in equations (5.2.1, 5.2.4).

Simply write $\mho_{\hat{Z}_{\omega,\beta}}$ as $\mho_{\hat{Z}}$. Define $\hat{w}_{\omega,\beta} := \hat{w}_{\hat{\sigma}} := -\Im\left(\overline{\langle \mho_{\hat{Z}}, \mathbf{v}\rangle_S} \cdot \mho_{\hat{Z}}\right)$ similarly. Bayer and Macrì already computed $\hat{w}_{\omega,\beta}$ when S is a K3 surface [BM14a, Lemma 9.2]. The corresponding computations are given in Appendix B.

3.3 Derived dual and correspondent line bundles

Recall the Corollary 2.6.2. Now $w_{\omega,\beta}(\mathrm{ch}) \in v(\mathrm{ch})^\perp$ and $w_{\omega,-\beta}(-\mathrm{ch}^*) \in v(-\mathrm{ch}^*)^\perp$. Let $\mathcal{E}$ be a universal family over $M_\sigma(\mathrm{ch})$. Denote $\mathcal{F}$ the dual universal family over $M_{\Phi(\sigma)}(-\mathrm{ch}^*)$.

Lemma 3.3.1. *Fix the Chern characters* ch. *Let* $\sigma := \sigma_{\omega,\beta}$ *and assume that* $Z_{\omega,\beta}(\mathrm{ch})$ *belongs to the open upper half plane (as Remark 2.6.3). Then* $\ell_\sigma \cong \ell_{\Phi(\sigma)}$, *i.e.* $\theta_{\sigma,\mathcal{E}}(w_{\omega,\beta}(\mathrm{ch})) \cong \theta_{\Phi(\sigma),\mathcal{F}}(w_{\omega,-\beta}(-\mathrm{ch}^*))$.

Proof. This is a consequence of isomorphism of the moduli spaces $M_{\sigma_{\omega,\beta}}(\mathrm{ch}) \cong M_{\sigma_{\omega,-\beta}}(-\mathrm{ch}^*)$ given by the duality functor $\Phi(\cdot) = R\mathcal{H}om(\cdot, \mathcal{O}_S)[1]$. $\square$

3.4 Identification of Néron-Severi groups

Let $\mathtt{C}$ be the chamber contains σ. The line bundle $\ell_{\sigma,\mathcal{E}}$ is only defined *locally*, i.e. $\ell_{\sigma,\mathcal{E}} \in N^1(M_{\mathtt{C}}(\mathrm{ch}))$. If we take another chamber $\mathtt{C}'$, we cannot say $\ell_{\sigma,\mathcal{E}} \in N^1(M_{\mathtt{C}'}(\mathrm{ch}))$ directly. We want to associate $\ell_{\sigma,\mathcal{E}}$ the *global* meaning in the following way.

Let $\sigma \in \mathtt{C}$ and $\tau \in \mathtt{C}'$ be two generic numerical stability conditions in different chambers for ch. *Assume* $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$ exist with universal families $\mathcal{E}$ and $\mathcal{F}$ respectively. And *assume* that there is a birational map between $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$, induced by a derived autoequivalence Ψ of $\mathrm{D}^{\mathrm{b}}(S)$ in the following sense: there exists a common open subset U of $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$, with complements of codimension at least two, such that for any $u \in U$, the corresponding objects $\mathcal{E}_u \in M_\sigma(\mathrm{ch})$ and $\mathcal{F}_u \in M_\tau(\mathrm{ch})$ are related by $\mathcal{F}_u = \Psi(\mathcal{E}_u)$. Then the Néron-Severi groups of $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$ can canonically be identified. So for a Mukai vector $w \in \mathbf{v}^\perp$, the two line bundles $\theta_{\mathtt{C},\mathcal{E}}(w)$ and $\theta_{\mathtt{C}',\mathcal{F}}(w)$ are identified.

If we can identify different Néron-Severi groups, we then associate $\ell_{\sigma,\mathcal{E}}$ the *global meaning*, i.e $\ell_{\sigma,\mathcal{E}} \in N^1(M_{\mathtt{C}'}(\mathrm{ch}))$ for any other chamber $\mathtt{C}'$. However, it is not nef on $M_{\mathtt{C}'}(\mathrm{ch})$.

Example 3.4.1. The above assumptions are satisfied if S is a K3 surface and $\mathbf{v} = v(\mathrm{ch})$ is primitive [BM14b, Theorem 1.1]. Bayer and Macrì then fixed a base stability condition $\sigma \in \mathrm{Stab}^\dagger(S)$ and glued the *local* Bayer-Macrì map to a *global* Bayer-Macrì map $\ell : \mathrm{Stab}^\dagger(S) \to N^1(M_\sigma(\mathrm{ch}))$. If $\mathrm{ch}_0 \neq 0$, the fixed stability condition is typically taking from the Gieseker chamber (see Definition 4.3.1), then $\ell_{\sigma,\mathcal{E}} \in N^1(M_{(\alpha,\omega)}(\mathrm{ch}))$.

Theorem 3.4.2. *[BM14b, Theorem 1.2] Let S be a smooth K3 surface. Fix a primitive Mukai vector $\mathbf{v}$. Fix a base point $\sigma \in \mathrm{Stab}^\dagger(S)$ in a chamber.*

(a). Under the identification of the Néron-Severi groups, there is a piece-wise analytic continuous map $\ell : \mathrm{Stab}^\dagger(S) \to N^1(M_\sigma(\mathrm{ch}))$.

(b). The image of ℓ is the intersection of the movable cone with the big cone of $M := M_\sigma(\mathbf{v})$: $\mathrm{Mov}(M) \cap \mathrm{Big}(M)$.

(c). The map ℓ is compatible, in the sense that for any generic $\sigma' \in \mathrm{Stab}^\dagger(S)$, the moduli space $M_{\sigma'}(\mathbf{v})$ is the birational model corresponding to $\ell(\sigma')$. In particular, every smooth K*-trivial birational model of M appears as a moduli space $M_{\mathtt{C}}(\mathbf{v})$ of Bridgeland stable objects for some chamber $\mathtt{C} \subset \mathrm{Stab}^\dagger(S)$.*

(d). For a chamber $\mathtt{C} \subset \mathrm{Stab}^\dagger(S)$, we have $\ell(\mathtt{C}) = \mathrm{Amp}(M_{\mathtt{C}}(\mathbf{v}))$.

$\mathrm{Mov}(M)$ has a locally polyhedral chamber decomposition. We call it the *Mori chamber decomposition* of $\mathrm{Mov}(M)$.

Example 3.4.3. Let M be a Mori dream space, and let M_i be the birational models in the Definition C.2.1. Then the Néron-Severi groups of different birational models M_i can be identified.

Chapter 4

Large volume limit and natural morphisms

This Chapter serves as the geometric background for the decomposition formulas in Chapter 5. We obtain some special Mukai vectors by taking limit of $w_{\omega,\beta}$ defined in equation (3.2.2). By applying Mukai morphism to those Mukai vectors, we get some interesting line bundles which have geometric meaning. We use Maciocia's notations in Section 2.5. The relation between the Bridgeland stability in the large volume limit and twisted Gieseker stability is recalled in Appendix A.

4.1 The semistable objects for $\omega^2 \gg 0$

Bridgeland [Bri08, Section 14], Arcara-Bertram [AB13, Section 3] and Arcara-Bertram-Coskun-Huizenga [ABCH13, Section 6] gave the description of stable object for $\omega^2 \gg 0$.

Lemma 4.1.1. *[ABCH13, Prop. 6.2] [BC13, Page 47] If $E \in \mathcal{A}_{tH,\beta}$ is stable for fixed β and $t \gg 0$, then we have the following two cases:*

- *case* $\mathrm{ch}_0 \geq 0$*: $E \in \mathcal{T}_{tH,\beta}$, or*
- *case* $\mathrm{ch}_0 < 0$*: $H^{-1}(E)$ is torsion free of rank $-\mathrm{ch}_0$, and $H^0(E)$ is a sheaf of finite length (or zero).*

Remark 4.1.2. In the second case, $H^{-1}(E)$ is actually locally free. If $H^{-1}(E)$ is not locally free, then the sheaf $(H^{-1}(E))^{**}/H^{-1}(E)$ would be a zero dimensional subobject of E and thus destabilizes it in the sense of Bridgeland stability.

4.2 Moduli space for $\omega^2 \gg 0$

Fix $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. Lo and Qin [LQ14] studied moduli spaces $M_{\sigma_{\omega,\beta}}(\mathrm{ch})$ for $\omega^2 \gg 0$. Recall the equation (A.0.6) $\alpha := \beta - \frac{1}{2}\mathrm{K}_S$. We list the following three cases according to the dimension of the support of objects with invariants ch. By Remark A.0.6, objects with $\mathrm{ch}_0 \geq 0$ in $\mathcal{A}_{\omega,\beta}$ are pure sheaves for $\omega^2 \gg 0$.

Supported in dimension 0

Suppose that $E \in \mathcal{A}_{\omega,\beta}$ is 0-dimensional and $\sigma_{\omega,\beta}$-semistable. Then $\mathrm{ch}(E) = (0, 0, n)$ for some positive integer n. Every object E with $\mathrm{ch}(E) = (0, 0, n)$ are σ-semistable with phase 1. And $t > 0$ is the only chamber in the (s, t)-half-plane.

Lemma 4.2.1. *[LQ14, Lemma 2.10] Fix* $\mathrm{ch} = (0, 0, n)$ *then*

$$M_{\sigma_{\omega,\beta}}(\mathrm{ch}) = M_{(\alpha,\omega)}(\mathrm{ch}) = \mathrm{Sym}^n(S) \quad \textit{for any } \beta.$$

Supported in dimension 1

Suppose that $E \in \mathcal{A}_{\omega,\beta}$ is 1-dimension and $\sigma_{\omega,\beta}$-semistable. Then $\mathrm{ch}_1(E).\omega > 0$. Recall the Simpson slope as in equation (A.0.5):

$$\xi_{\omega,\beta}(E) := \frac{\mathrm{ch}_2(E) - \mathrm{ch}_1(E).\beta}{\omega.\mathrm{ch}_1(E)}.$$

Note that $\xi_{tH,\beta}(F) \leq \xi_{tH,\beta}(E)$ is equivalent to $\xi_{H,\beta}(F) \leq \xi_{H,\beta}(E)$ for $t > 0$. For a fixed β, $\sigma_{tH,\beta}$ induce the fixed heart $\mathcal{A}_{H,\beta}$ (see Remark 2.1.4) and equivalent central charge $Z_{tH,\beta}$ for $t \gg 0$.

Lemma 4.2.2. *[LQ14, Lemma 2.11] Let* $\alpha = \beta - \frac{1}{2}\mathrm{K}_S$. *Fix* $\mathrm{ch} = (0, \mathrm{ch}_1, \mathrm{ch}_2)$ *with* $\mathrm{ch}_1.H > 0$ *and let* $t \gg 0$, *then*

$$M_{\sigma_{\omega,\beta}}(\mathrm{ch}) = M_{(\alpha,\omega)}(\mathrm{ch}) \quad \textit{for any } \beta.$$

And the (α, ω)-Gieseker semistability is the Simpson semistability defined by $\xi_{H,\beta}$ in equation (A.0.5).

It is clear that E in above moduli space is pure 1-dimensional sheaf. Otherwise the non-pure part is 0-dimensional and would destabilize E in the sense of Bridgeland stability.

Supported in dimension 2

The nonzero rank cases are classified by Lo-Qin in the following three cases. We refer them as the Gieseker chamber case, the dual Gieseker chamber case, and the dual Uhlenbeck wall case.

Assumption 4.2.3. An invariant $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ satisfies *condition* (C) if the following three assumptions holds:

- $\mathrm{ch}_0 > 0$;
- (Bogomolov) $\mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2 \geq 0$;
- $\gcd(\mathrm{ch}_0, \mathrm{ch}_1.H, \mathrm{ch}_2 - \frac{1}{2}\mathrm{ch}_1.\mathrm{K}_S) = 1$ for a fixed ample line bundle H ([HL10, Corollary 4.6.7]);

An object E satisfies *condition* (C) if $\mathrm{ch}(E)$ satisfies condition (C).

These conditions guarantee the existence of the fine moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ ([HL10, Corollary 4.6.7] and Remark 5.4.5).

Lemma 4.2.4. *[Bri08] [LQ14, Lemma 5.1]* (Gieseker chamber) *Let* $\alpha = \beta - \frac{1}{2}\mathrm{K}_S$. *Fix* $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ *with* $\mathrm{ch}_0 > 0$. *Assume that* ch *satisfies condition* (C) *as in Assumption 4.2.3. Let* $t \gg 0$ *and* $s < s_0 := \frac{\mathrm{ch}_1.H}{\mathrm{ch}_0 H^2}$, *then*

$$M_{\sigma_{\omega,\beta}}(\mathrm{ch}) \cong M_{(\alpha,\omega)}(\mathrm{ch}).$$

Lemma 4.2.5. *[LQ14, Theorem 5.4]* (Dual Uhlenbeck wall) *Fix* $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ *with* $\mathrm{ch}_0 < 0$. *Assume that* $-(\mathrm{ch})^*$ *satisfies condition* (C) *as in Assumption 4.2.3. Let* $t \gg 0$ *and* $s = s_0$, *then*

$$M_{\sigma_{\omega,\beta}}(\mathrm{ch}) \cong U_\omega(-(\mathrm{ch})^*),$$

where $U_\omega(-(\mathrm{ch})^*) = U_H(-(\mathrm{ch})^*)$ is the Uhlenbeck compactification of the moduli space $M_\omega^{\mathrm{lf}}(-(\mathrm{ch})^*)$ of locally free sheaves with invariant $-(\mathrm{ch})^*$.

4.3 Trivial chamber, Simpson chamber, (dual) Gieseker chamber

Definition 4.3.1. Fix $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. Fix a Maciocia's coordinate (H, γ, u) and consider Bridgeland stability conditions on the (s,t)-half-plane $\Pi_{(H,\gamma,u)}$. Recall $s_0 = \frac{\mathrm{ch}_1.H}{\mathrm{ch}_0 H^2}$ if $\mathrm{ch}_0 \neq 0$.

`TC` If $\mathrm{ch} = (0, 0, n)$ with positive integer n as in Lemma 4.2.1, then there is *no wall*, and $t > 0$ is the *trivial chamber* in $\Pi_{(H,\gamma,u)}$.

`SC` If $\mathrm{ch}_0 = 0$, and $\mathrm{ch}_1.H > 0$ as in Lemma 4.2.2, we define the chamber for $t \gg 0$ as the *Simpson chamber with respect to* (H, γ, u). In this case, β is arbitrary.

`GC` If $\mathrm{ch}_0 > 0$, as in Lemma 4.2.4, we define the chamber for $t \gg 0$ and $s < s_0$ as the *Gieseker chamber with respect to* (H, γ, u).

`UW` If $\mathrm{ch}_0 > 0$, as in Lemma 4.2.4, we define the wall $t > 0$ and $s = s_0$, i.e. $\Im Z(\mathrm{ch}) = 0$ as the *Uhlenbeck wall with respect to* (H, γ, u).

`DGC` If $\mathrm{ch}_0 < 0$, we define the chamber for $t \gg 0$ and $s > s_0$ as the *dual Gieseker chamber with respect to* (H, γ, u).

`DUW` If $\mathrm{ch}_0 < 0$, as in Lemma 4.2.5, we define the wall $t > 0$ and $s = s_0$, i.e. $\Im Z(\mathrm{ch}) = 0$ as the *dual Uhlenbeck wall with respect to* (H, γ, u). We extend the definition in Proposition-Definition 4.4.1.

When H is clear from the context, we simply omit the H. We refer the Figure 4.1.

Remark 4.3.2. The definition of `UW` essentially means that every object with $\mathrm{ch}_0 > 0$ is of phase 0 for $\sigma \in$ `UW`. The original definition of Bridgeland stability condition $\sigma = (Z, \mathcal{P}(\phi))$ [Bri07] consists of a group homomorphism $Z : K(\mathrm{D}) \to \mathbb{C}$, together with a slicing $\mathcal{P}$ of the triangulated category D such that if $0 \neq E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$ then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$. The heart of the t-structure $\mathcal{P}(>\phi)$ is the abelian category $\mathcal{P}((\phi,\phi+1])$, and the heart of the t-structure $\mathcal{P}(\geq\phi)$ is the abelian category $\mathcal{P}([\phi,\phi+1))$. The definition in Chapter 2 is the former choice $\mathcal{A} := \mathcal{P}((0,1])$, which is also called *right* Bridgeland stability condition. By definition $\sigma_{\omega,\beta}$ is a right stability condition. There is a notion of *left* Bridgeland stability condition for the heart $\mathcal{P}([0,1))$ [Oka06, Remark 3.5]. So $\sigma \in \mathtt{UW}$ is not a right stability condition, but a left stability condition. It is interesting to give a description of heart for $\sigma \in \mathtt{UW}$.

Remark 4.3.3. We continue the Remark 4.1.2. Let $G := H^{-1}(E)$ be the *locally free* sheaf, and $Q := H^0(E)$ be the 0-dimensional sheaf. We have the canonical sequence

$$0 \longrightarrow G[1] \longrightarrow E \longrightarrow Q \longrightarrow 0 \quad \text{in} \quad \mathcal{A}_{\sigma_{tH,s>s_0}}, \tag{4.3.1}$$

for stability $\sigma_{tH,s>s_0} \in \mathtt{DGC}$ or $\sigma_{tH,s=s_0} \in \mathtt{DUW}$. The notation $\sigma_{tH,s>s_0}$ means $\sigma_{tH,sH+u\gamma} \in \Pi_{(H,\gamma,u)}$, with $s > s_0$. Let $F := E^\vee[1]$, so $E = F^\vee[1]$. Recall that for the 0-dimensional sheaf Q, we have $Q^\vee = P[-2]$ for a 0-dimensional sheaf P [Huy06, Corollary 3.40].

- For $\sigma_{tH,s>s_0} \in \mathtt{DGC} \subset \Pi_{(H,\gamma,u)}$, by applying $(\cdot)^\vee[1]$ to the sequence (4.3.1), we obtain an exact triangle $P[-1] \longrightarrow F \longrightarrow G^\vee \xrightarrow{[1]} P$. Since G is a locally free sheaf, we have $H^i(G^\vee) = \mathcal{E}xt^i(G,\mathcal{O}_S) = 0$ for $i \neq 0$ [HL10, Proposition 1.1.10]. So $G^\vee$ is a sheaf. Moreover $G^\vee = H^0(G^\vee) = \mathcal{H}om(G,\mathcal{O}_S) = G^*$. The subobject F of $G^\vee$ must be a sheaf [AB13, Lemma 3.4]. So $G = H^{-1}(E) = H^0(F^\vee) = \mathcal{H}om(F,\mathcal{O}_S) = F^*$. Since F^* is locally free, we still have $(F^*)^\vee = F^{**}$. Combining above statements, we obtain $G^\vee = (F^*)^\vee = F^{**}$. And there is an exact sequence

$$0 \longrightarrow F \longrightarrow F^{**} \longrightarrow P \longrightarrow 0 \quad \text{in} \quad \mathcal{A}_{\sigma_{tH,s<-s_0}},$$

  for $\sigma_{tH,s<-s_0} \in \mathtt{GC} \subset \Pi_{(H,\gamma,-u)}$.

- For $\sigma_{tH,s=s_0} \in \mathtt{DUW}$, we have

$$E \xlongequal{S-\text{equi.}} G[1] \oplus Q, \tag{4.3.2}$$

  and all objects are semistable of phase 1 in the heart $\mathcal{A}_{tH,s=s_0}$. We cannot directly apply $(\cdot)^\vee[1]$ to the equation (4.3.2). Otherwise, we would have $F = E^\vee[1] \xlongequal{S-\text{equi.}} F^{**} \oplus P[-1]$. However, $P[-1]$ has phase 0 and will never be in a heart of Bridgeland stability condition. Similarly we cannot take $t = +\infty$ in the Gieseker chamber. The hidden structure will be clear in Proposition 4.5.1.

4.4 Simpson wall, (dual) Uhlenbeck wall

Fix a Maciocia's coordinate (H, γ, u). Let us consider the limit behavior $t = +\infty$.

- For $\sigma_{tH,sH+u\gamma} \in \mathtt{SC}$, we *can* take $t = +\infty$ since there is no restriction for the heart (or the β).
- For $\sigma_{tH,sH+u\gamma} \in \mathtt{GC}$, we *cannot* take $t = +\infty$. Otherwise, by using equation (2.1.1), every object would have phase $0 \notin (0, 1]$.
- For $\sigma_{tH,sH+u\gamma} \in \mathtt{DGC}$, we *can* take $t = +\infty$, and every object has phase 1.
- For $\sigma_{tH,s_0H+u\gamma} \in \mathtt{DUW}$, note that t is any positive number. In particular, we *can* take $t = +\infty$. The heart will not change and every object has phase 1.

Let us compute the limit cases of $w_{\omega,\beta}$ in Lemma 3.2.2. We only care about $w_{\omega,\beta}$ up to a positive scalar.

- Let $\sigma_{tH,sH+u\gamma} \in \mathtt{SC}$. In this case, $\mathrm{ch}_0 = 0$ and $\mathrm{ch}_1.H > 0$. By using equation (2.1.2), we have $\Im Z(\mathrm{ch}) = t\mathrm{ch}_1.H > 0$ and $\Re Z(\mathrm{ch}) = -\mathrm{ch}_2 + \mathrm{ch}_1.\beta$. We can take $t = +\infty$ and obtain

$$w_{\infty H,\beta} \overset{\mathbb{R}_+}{=\!=} (0, 0, -1).$$

- Let $\sigma_{tH,sH+u\gamma} \in \mathtt{DGC}$. In this case, $\mathrm{ch}_0 < 0$. We can take $t = +\infty$ and obtain

$$w_{\infty H,\beta} \overset{\mathbb{R}_+}{=\!=} (0, H, (\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \frac{3}{4}\mathrm{K}_S)H). \tag{4.4.1}$$

- Let $\sigma_{tH,s_0H+u\gamma} \in \mathtt{DUW}$. In this case, $\mathrm{ch}_0 < 0$ and $s = s_0 = \frac{\mathrm{ch}_1.H}{\mathrm{ch}_0 H^2}$. Since $\Im Z(\mathrm{ch}) = 0$ (this is the reason why we take $s = s_0$), we have $\Re Z(\mathrm{ch}) < 0$ and obtain

$$w_{\sigma\in\mathtt{DUW}} \overset{\mathbb{R}_+}{=\!=} (0, H, (\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \frac{3}{4}\mathrm{K}_S)H). \tag{4.4.2}$$

Proposition-Definition 4.4.1. We extend the definition of walls on the (s,t)-half-plane $\Pi_{(H,\gamma,u)}$ to the case $t = +\infty$. Recall $s_0 = \frac{\mathrm{ch}_1 H}{\mathrm{ch}_0 H^2}$ if $\mathrm{ch}_0 \neq 0$.

DUW Fix ch with $\mathrm{ch}_0 < 0$. Since equations (4.4.1) and (4.4.2) are the same, we define the *dual Uhlenbeck wall* DUW with respect to (H, γ, u) as the vertical half-line $(s = s_0, t > 0)$ together with the horizontal half-line $(s \geq s_0, t = +\infty)$. In particular,

$$w_{\sigma \in \mathtt{DUW}} \xlongequal{\mathbb{R}_+} (0, H, (\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \frac{3}{4}\mathrm{K}_S)H) =: \mathbf{m}(H, \mathrm{ch}), \tag{4.4.3}$$

and $\mu_{\sigma \in \mathtt{DUW}}(\mathrm{ch}) = -\frac{\Re Z(\mathrm{ch})}{\Im Z(\mathrm{ch})} = +\infty$. Every object with invariants ch is σ-semistable of phase 1 for $\sigma \in \mathtt{DUW}$.

UW Fix ch with $\mathrm{ch}_0 > 0$. We define the *Uhlenbeck wall* UW with respect to (H, γ, u) as the vertical half-line $(s = s_0, t > 0)$ together with the horizontal half-line $(s < s_0, t = +\infty)$. Every object with invariants ch is of phase 0 for $\sigma \in \mathtt{UW}$. Stability conditions on UW are left stability conditions (Remark 4.3.2).

SW Fix ch with $\mathrm{ch}_0 = 0$ and $\mathrm{ch}_1.H > 0$. We define the *Simpson wall* SW with respect to (H, γ, u) by the horizontal line $t = +\infty$. Every object with invariants ch is σ-semistable of phase $\frac{1}{2}$ for $\sigma \in \mathtt{SW}$. In particular,

$$w_{\sigma \in \mathtt{SW}} \xlongequal{\mathbb{R}_+} (0, 0, -1). \tag{4.4.4}$$

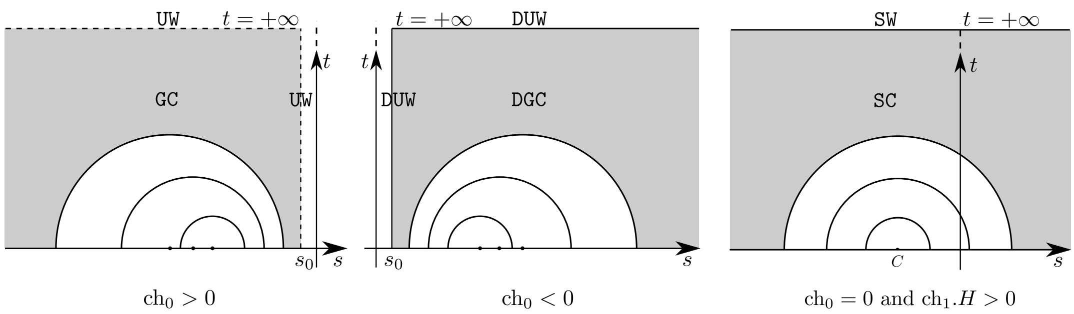


Figure 4.1: Wall-chamber structures in the halfplane $\Pi_{(H,\gamma,u)}$ in three cases

In summary, the dual Uhlenbeck wall can be touched from the dual Gieseker chamber, i.e. we can move a stability condition in DGC continuously to a stability condition at DUW. However, the Uhlenbeck wall *cannot* be touched from the Gieseker chamber. The Simpson wall SW can be touched from the Simpson chamber SC.

4.5 Gieseker-Uhlenbeck morphism

Proposition 4.5.1. *Assume that* ch *satisfies the condition* (C) *as in Assumption 4.2.3. Moving stability conditions σ from the dual Gieseker chamber to the dual Uhlenbeck wall, there is a natural morphism induced by the nef line bundle $\widetilde{H}$:*

$$M_{\sigma\in\mathtt{DGC}}(-\mathrm{ch}^*) \xrightarrow{\widetilde{H}} M_{\sigma\in\mathtt{DUW}}(-\mathrm{ch}^*), \tag{4.5.1}$$

where $\widetilde{H} := \theta_{\mathtt{DGC},\mathcal{F}}(\mathbf{m}(H, -\mathrm{ch}^*))$. Moreover, this morphism can be identified with the Gieseker-Uhlenbeck morphism.

Proof. With the assumption (C), the moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ exists. So by Lemma 2.6.1 and Lemma 4.2.4, $M_{\sigma\in\mathtt{DGC}}(-\mathrm{ch}^*) \overset{\Phi}{=\!=} M_{\sigma\in\mathtt{GC}}(\mathrm{ch}) = M_{(\alpha,\omega)}(\mathrm{ch})$ exists as a projective scheme. Denote $\mathcal{E}$ a universal family over $M_{\sigma\in\mathtt{GC}}(\mathrm{ch})$ and denote $\mathcal{F}$ the dual universal family over $M_{\sigma\in\mathtt{DGC}}(-\mathrm{ch}^*)$. By the work of J. Li [Li93], there is the Gieseker-Uhlenbeck morphism

$$\pi : M_{(\alpha,\omega)}(\mathrm{ch}) \xrightarrow{\mathcal{L}_1} U_\omega(\mathrm{ch}). \tag{4.5.2}$$

Moreover, such morphism is induced by $\mathcal{L}_1 := \lambda(u_1(\mathrm{ch}))$ [HL10, Theorem 8.2.8]. We postpone the precise definition of $\mathcal{L}_1$ to Section 5.3. By Proposition 5.3.2, we have $\mathcal{L}_1 = \mathrm{ch}_0\widetilde{H}$. The existence of the morphism (4.5.1) follows from the diagram

$$\begin{tikzcd}
M_{(\alpha,\omega)}(\mathrm{ch}) \arrow[r, "\mathcal{L}_1", "\mathrm{GU}"'] \arrow[d, equal, "\text{Large volumn limit}"', "\cong"] & U_\omega(\mathrm{ch}) \arrow[dd, equal, "\mathrm{LQ}"', "\cong"] \\
M_{\sigma\in\mathtt{GC}}(\mathrm{ch}) \arrow[d, equal, "\Phi=(\cdot)^\vee[1]"', "\cong"] & \\
M_{\sigma\in\mathtt{DGC}}(-\mathrm{ch}^*) \arrow[r, "\widetilde{H}", "\mathrm{BM}"'] & M_{\sigma\in\mathtt{DUW}}(-\mathrm{ch}^*)
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, mapsto, "\mathrm{GU}"'] \arrow[d, equal] & (F^{**}, P) \arrow[dd, equal, "\mathrm{LQ}"', "\cong"] \\
F = E^\vee[1] \arrow[d, equal, "\Phi=(\cdot)^\vee[1]"', "\cong"] & \\
E = F^\vee[1] \arrow[r, mapsto, "\mathrm{BM}"'] & [G[1] \oplus Q]
\end{tikzcd}$$

where we use Lemma 4.2.5 for the right vertical isomorphism.

Let E be an object in $M_{\sigma\in\mathtt{DGC}}(-\mathrm{ch}^*)$, then $F := E^\vee[1]$ is the corresponding object in $M_{\sigma\in\mathtt{GC}}(\mathrm{ch})$ (Remark 4.3.3). $G := H^{-1}(E)$ is a locally free sheaf. $Q := H^0(E)$ is a 0-dimensional sheaf (or the zero sheaf when the GU morphism is an isomorphism). $P = Q^\vee[2]$ is also a 0-dimensional sheaf. There is an exact sequence of sheaves $0 \to F \to F^{**} \to P \to 0$. There is also an exact sequence $0 \to G[1] \to E \to Q \to 0$ in the heart for DGC. At the DUW this exact sequence gives the

S-equivalent relation $E \xlongequal{S-\text{equi.}} G[1]\oplus Q$. $[G[1]\oplus Q]$ means its S-equivalence class. The union of moduli spaces parametrizing (F^{**}, P) is a stratification $U_\omega(\mathrm{ch}) = \sqcup_{l\geq 0} M^{\mathrm{lf}}_\omega(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2 + l) \times S^{(l)}$, where $F^{**} \in M^{\mathrm{lf}}_\omega(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2 + l)$ and $P \in S^{(l)}$. □

The original construction of Uhlenbeck compactification $U_\omega(\mathrm{ch})$ is in an analytic way, which is a compactification of the space of gauge equivalence classes of irreducible ASD connections on the SU(2) principal bundle E with fixed topological type ch. By Donaldson's results, such irreducible ASD connection on E induces a holomorphic structure on E which is Mumford stable. On the other hand, J. Li constructed a morphism from $M_{(\alpha,\omega)}(\mathrm{ch})$ to a projective space, and the image scheme of $M_{(\alpha,\omega)}(\mathrm{ch})$ under this morphism is homeomorphic to the (analytic) Uhlenbeck compactification $U_\omega(\mathrm{ch})$. Therefore there is a complex structure on $U_\omega(\mathrm{ch})$ making it a reduced projective scheme. When we mention $U_\omega(\mathrm{ch})$ in the GU morphism, we always mean it in the sense of J. Li's algebraic construction. $U_\omega(\mathrm{ch})$ has a set-theoretically stratification. Lo and Qin identified $M_{\sigma\in\mathtt{DUW}(-\mathrm{ch}^*)}$ with $U_\omega(\mathrm{ch})$ set-theoretically (Lemma 4.2.5). Lo [Lo12, Theorem 3.1] further identified them as moduli space in the case of K3 surfaces by using the Fourier-Mukai transforms and Huybrechts' description of the tilted hearts of the K3 surfaces [Huy08].

Problem 4.5.2. *It is interesting to give a direct proof of the existence of morphism (4.5.1) without using J. Li's construction. It is also interesting to give a similar proof of the existence of support morphism (Conjecture 4.6.3). One difficulty is a direct construction of the moduli scheme* $M_{\sigma\in\mathtt{DUW}}(-\mathrm{ch}^*)$.

The boundary of $M_{(\alpha,\omega)}(\mathrm{ch})$ is the set

$$\partial M_{(\alpha,\omega)}(\mathrm{ch}) := \{m \in M_{(\alpha,\omega)}(\mathrm{ch}) |\ \mathcal{E}_m \text{ is not locally free}\}.$$

If $\partial M_{(\alpha,\omega)}(\mathrm{ch}) = \emptyset$, then every objects $F \in M_{(\alpha,\omega)}(\mathrm{ch})$ is locally free and $F \cong F^{**}$. So the Gieseker-Uhlenbeck morphism π is an isomorphism. In the following, we always *assume* that $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$. Taking $m \in \partial M_{(\alpha,\omega)}(\mathrm{ch})$, we have an exact sequence of sheaves $0 \to \mathcal{E}_m \to (\mathcal{E}_m)^{**} \to P \to 0$, for some 0-dimensional nonzero sheaf P. So the boundary is exactly the locus contracted by the Gieseker-Uhlenbeck morphism.

Lemma 4.5.3. *[HL10, Lemma 9.2.1]* $\partial M_{(\alpha,\omega)}(\mathrm{ch})$ *is a closed subset of* $M_{(\alpha,\omega)}(\mathrm{ch})$, *and if* $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$, *then* $\mathrm{codim}(\partial M_{(\alpha,\omega)}(\mathrm{ch}), M_{(\alpha,\omega)}(\mathrm{ch})) \leq \mathrm{ch}_0 - 1$.

In particular, if $\mathrm{ch}_0 = 2$ and $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$, then the Gieseker-Uhlenbeck morphism is a divisorial contraction.

4.6 Support morphism

In previous section, we know the moduli space $M_{\sigma\in\mathtt{DUW}}(-\mathrm{ch}^*)$ is a projective scheme by identifying it with the Uhlenbeck space $U_H(\mathrm{ch})$. In this section, we fix ch with $\mathrm{ch}_0 = 0$, and assume that $\mathrm{ch}_1.H > 0$. We already know that $M_{\sigma\in\mathtt{SC}}(\mathrm{ch})$ is projective scheme by identifying it with the Simpson moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ which parametrizes pure 1 dimensional sheaves with numerical type $\mathrm{ch} = (0, \mathrm{ch}_1, \mathrm{ch}_2)$. However, we don't know whether $M_{\sigma\in\mathtt{SW}}(\mathrm{ch})$ is projective scheme. We will give the description of $M_{\sigma\in\mathtt{SW}}(\mathrm{ch})$ set-theoretically. We would ask three natural questions:

- Q1: Is the moduli space $M_{\sigma\in\mathtt{SW}}(\mathrm{ch})$ a projective scheme?
- Q2: If $M_{\sigma\in\mathtt{SW}}(\mathrm{ch})$ a projective scheme, is there a natural morphism $M_{\sigma\in\mathtt{SC}}(\mathrm{ch}) \to M_{\sigma\in\mathtt{SW}}(\mathrm{ch})$?
- Q3: If there is a natural morphism $M_{\sigma\in\mathtt{SC}}(\mathrm{ch}) \to M_{\sigma\in\mathtt{SW}}(\mathrm{ch})$, which line bundle on $M_{\sigma\in\mathtt{SC}}(\mathrm{ch})$ induces such morphism?

We first give some preliminaries. It is easy to see $(0, 0, -1) \in \mathbf{v}^{\perp}$. So we obtain a line bundle

$$\mathcal{S} := \theta_{\mathtt{SC},\mathcal{E}}((0, 0, -1)) \in N^1(S). \tag{4.6.1}$$

By Corollary 2.6.2, the duality functor $(\cdot)^{\vee}[1]$ gives the isomorphism

$$M_{\sigma_{tH,\beta}}((0, \mathrm{ch}_1, \mathrm{ch}_2)) \cong M_{\sigma_{tH,-\beta}}((0, \mathrm{ch}_1, -\mathrm{ch}_2)) \text{ for } t > 0.$$

The above questions can be answered in special cases.

Example 4.6.1. (Woolf [Woo13]). Let $S = \mathbb{P}^2$. Denote $N(\mu, \chi)$ the moduli space of Simpson semistable pure 1-dimensional sheaves on $\mathbb{P}^2$ with Hilbert polynomial $\chi(E(m)) = \mu m + \chi$. Denote $H \in \mathbb{P}^2$ the hyperplane divisor. Then $N(\mu,\chi) = M_{\sigma\in\mathtt{SC}}(\mathrm{ch})$, with $\mathrm{ch} = (0, \mu H, \chi - \frac{3}{2}\mu)$. $N(\mu,\chi)$ is irreducible $\mathbb{Q}$-factorial variety of dimension μ^2+1 with only canonical singularities. There is an isomorphism $N(\mu,\chi) \cong N(\mu,\chi+\mu)$ by the map of tensoring line bundle $E \mapsto E \otimes \mathcal{O}_{\mathbb{P}^2}(1)$. The duality functor $(\cdot)^\vee[1]$ gives the isomorphism $N(\mu,\chi) \cong N(\mu,-\chi)$. Woolf identified $M_{\sigma\in\mathtt{SW}}(\mathrm{ch})$ with $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(\mu)))$ and gave the *support morphism* $f : N(\mu,\chi) \to \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(\mu)))$, which maps a pure 1 dimensional sheaf E to its support $\mathrm{Supp}(E)$, and the scheme-structure on $\mathrm{Supp}(E)$ is given by the Fitting ideal of E. Moreover the morphism is induced by the line bundle $\mathcal{S}$ (notation $\mathcal{L}_0$ therein). The dimension of $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(\mu)))$ is $\binom{\mu+2}{2} - 1$. The fiber dimension of f is $\binom{\mu-1}{2}$. The morphism f is an isomorphism if $\mu = 1$ or 2. So we assume that $\mu \geq 3$ and $0 \leq \chi \leq \frac{\mu}{2}$. $\mathcal{S}$ is nef on $N(\mu,\chi)$, and forms one boundary of nef cone $\mathrm{Nef}(N(\mu,\chi))$. The Picard number of $N(\mu,\chi)$ is 2. The other boundary of $\mathrm{Nef}(N(\mu,\chi))$ will be given in Example 5.4.2.

Example 4.6.2. This example is given by Bayer and Macrì [BM14b, Section 11] and plays an important role in the proof of the existence of Lagrangian fibrations [BM14b, Theorem 1.5]. Let Y be a smooth projective K3 surface. Fix $\mathrm{ch} = (0, C, s)$, with $C \in \mathrm{Pic}(Y)$, $C.C > 0$ and $s \in \mathbb{Z}$. Let $H' \in \mathrm{Amp}(Y)$ be a generic polarization for ch. Then the (0-twisted) moduli space $M_{H'}(\mathrm{ch})$ admits a structure of Lagrangian fibration induced by global sections of $\mathcal{S}$. In this case the $M_{\sigma\in\mathtt{SW}}(\mathrm{ch}) \cong \mathbb{P}^m$ with $m = \frac{C.C+2}{2}$, the half dimension of $M_{H'}(\mathrm{ch})$.

We observed that

- every object $E \in M_{\sigma\in\mathtt{SC}}(\mathrm{ch})$ is a pure sheaf of dimension 1;
- every object with invariants ch is σ-semistable of phase $\frac{1}{2}$ for $\sigma \in \mathtt{SW}$;
- [HL10, Proposition 1.5.2] Jordan-Hölder filtrations always exist, i.e. every object $E \in M_{\sigma\in\mathtt{SC}}(\mathrm{ch})$ has a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E, \tag{4.6.2}$$

  such that the factors $\mathrm{gr}_i(E) := E_i/E_{i-1}$ are stable with the reduced Hilbert polynomial $p(E)$. Up to isomorphism, the sheaf $\mathrm{gr}(E) := \oplus_i \mathrm{gr}_i(E)$ does not depend on the choice of the Jordan-Hölder filtration.

Set-theoretically, $M_{\sigma\in\mathtt{SW}}(\mathrm{ch})$ is the set of S-equivalent classes of pure 1-dimensional sheaves $E \in M_{\sigma\in\mathtt{SC}}(\mathrm{ch})$. We then have a contraction map $M_{\sigma\in\mathtt{SC}}(\mathrm{ch}) \xrightarrow{\mathcal{S}} M_{\sigma\in\mathtt{SW}}(\mathrm{ch})$ set-theoretically, which maps $E \in M_{\sigma\in\mathtt{SC}}(\mathrm{ch})$ to $\mathrm{gr}(E)$. And $\mathrm{gr}(E)$ represents the S-equivalent class of E. Based on above examples, we have the following conjecture.

Conjecture 4.6.3. *Fix* ch *with* $\mathrm{ch}_0 = 0$*, and assume that* $\mathrm{ch}_1.H > 0$*. Assume the fine moduli space of pure* 1*-dimensional sheaves* $M_{(\alpha,\omega)}(\mathrm{ch})$ *exists where* $\alpha = \beta - \frac{1}{2}\mathrm{K}_S$ *as equation (A.0.6). Then there is a natural morphism (called support morphism)*

$$M_{\sigma\in\mathtt{SC}}(\mathrm{ch}) \xrightarrow{\mathcal{S}} M_{\sigma\in\mathtt{SW}}(\mathrm{ch}), \tag{4.6.3}$$

induced by the nef line bundle

$$\mathcal{S} := \theta_{\mathtt{SC},\mathcal{E}}((0,0,-1)). \tag{4.6.4}$$

4.7 Classical twisted Gieseker wall-chambers

By Remark A.0.7, we vary the (α, H) in the classical twisted Gieseker wall-chambers. Assume that $(\alpha, H)_0$ is on a wall, and $(\alpha, H)_\pm$ are in the adjacent different chambers. Then the following classical results can be understood by changing the Bridgeland stabilities and using Bayer-Macrì's theory:

$$\begin{tikzcd}
M_{(\alpha,H)_-}(\mathrm{ch}) \arrow[r, "\mathrm{BM}"] \arrow[d, "\mathrm{GU}-(H)_-"] & M_{(\alpha,H)_0}(\mathrm{ch}) \arrow[d, "\mathrm{GU}-(H)_0"] & M_{(\alpha,H)_+}(\mathrm{ch}) \arrow[l, "\mathrm{BM}"'] \arrow[d, "\mathrm{GU}-(H)_+"] \\
U_{(H)_-}(\mathrm{ch}) \arrow[r, "\mathrm{HL}"] & U_{(H)_0}(\mathrm{ch}) & U_{(H)_+}(\mathrm{ch}) \arrow[l, "\mathrm{HL}"']
\end{tikzcd}$$

where the last line was constructed by Hu and Li [HL95]. The notation $(H)_0$ means the ample line bundle in the pair $(\alpha, H)_0$. We have similar notations for $(H)_\pm$. Since $(\alpha, H)_0$ is a wall in the twisted Gieseker wall-chamber structures, $(H)_0$, $(H)_+$ and $(H)_-$ could be *equal*. Yoshioka [Yos14, Section 5.2] obtained the same results. The wall-chamber structures in the large volume limit are summarized in Table 4.1.

Table 4.1: Large volume limit: Definition A.0.1

Fix ch	LVL(ch) (A.0.1)	LVL(ch, H) for fixed $H \in \mathrm{Amp}(S)$ (A.0.2)	Objects
$\mathrm{ch}_0 > 0$	classical (α, ω)-Gieseker wall-chambers	Gieseker chamber for ch w.r.t. H	torsion free sheaves
$\mathrm{ch}_0 < 0$	Bridgeland wall-chambers	Dual Gieseker chamber for ch w.r.t. H	$H^{-1}(E)$ locally free, $H^0(E)$ supports in dim. 0
$\mathrm{ch}_0 = 0$ and $\mathrm{ch}_1.\omega > 0$ for $\omega \in \mathrm{Amp}(S)$	classical (α, ω)-Gieseker wall-chambers	Simpson chamber for ch w.r.t. H	pure sheaves of dim. 1
$\mathrm{ch} = (0, 0, n)$	no wall	trivial chamber $t > 0$	pure sheaves of dim. 0

Chapter 5

Bayer-Macrì decomposition

Let $\sigma \in \mathrm{Stab}^{\dagger}(S)$ as in Remark 2.4.2. We can assume $\sigma = \sigma_{\omega,\beta}$ after a group action. The Bayer-Macrì nef line bundle is constructed by $\ell_{\sigma_{\omega,\beta}} \stackrel{\mathbb{R}_+}{=\!=} \theta_{\sigma,\mathcal{E}}(w_{\omega,\beta})$, where $w_{\omega,\beta}$ has a preliminary decomposition as in Lemma 3.2.2. However, we cannot apply the Mukai morphism $\theta_{\sigma,\mathcal{E}}$ directly to those components because none of them is in $\mathbf{v}^{\perp}$. In this Chaper, we give an *intrinsic* decomposition of the Mukai vector $w_{\omega,\beta}$ in Theorem 5.2.1 and Theorem 5.2.4. In particular, each components is in $\mathbf{v}^{\perp}$. So we can apply $\theta_{\sigma,\mathcal{E}}$ and obtain the *intrinsic* decomposition of $\ell_{\sigma_{\omega,\beta}}$. We call such intrinsic decomposition of $w_{\omega,\beta}$ or $\ell_{\sigma_{\omega,\beta}}$ as the *Bayer-Macrì decomposition.*

The parallel results by using $\hat{\sigma}_{\omega,\beta}$ and $\Omega_{\hat{Z}}$ for K3 surfaces are given in Appendix B.

5.1 Preliminary computation

Let us compute the following two terms which appear in Lemma 3.2.2 by using Maciocia's notation. The computation will be used later.

Lemma 5.1.1. *Fix a choice of Maciocia's coordinate* (H, γ, u) *(Definition 2.5.1). We have the following two relations:*

$$\mu_\sigma(\mathrm{ch})\omega + \beta = C(\mathrm{ch}, \mathrm{ch}')H + u\gamma; \tag{5.1.1}$$

$$\beta.[\mu_\sigma(\mathrm{ch})\omega + \beta] - \frac{1}{2}(\omega^2 + \beta^2) = -\frac{g}{2}D(\mathrm{ch}, \mathrm{ch}') - \frac{d}{2}u^2, \tag{5.1.2}$$

where the numbers $C(\mathrm{ch}, \mathrm{ch}')$ *and* $D(\mathrm{ch}, \mathrm{ch}')$ *are given by equations (2.5.4, 2.5.5).*

Proof. The proof is a direct computation by using Maciocia's Theorem 2.5.2. For the reader's convenience, we give the details. For the equation (5.1.1), we only need to check that

$$\mu_\sigma(\mathrm{ch})t + s = C(\mathrm{ch}, \mathrm{ch}').$$

Recall the wall equation is $(s-C)^2 + t^2 = D + C^2$. Now

$$\begin{aligned}
\mu_\sigma(\mathrm{ch})t + s &= \frac{z - sy_1g + uy_2d + \frac{x}{2}(s^2g - u^2d - t^2g)}{(y_1 - xs)g} + s \\
&= \frac{z + uy_2d - \frac{x}{2}u^2d - \frac{xg}{2}(s^2+t^2)}{(y_1 - xs)g} \\
&= \frac{z + uy_2d - \frac{x}{2}u^2d - \frac{xg}{2}(2sC+D)}{(y_1 - xs)g} \qquad \text{by using wall equation.}
\end{aligned}$$

So we only need to check that

$$z + uy_2d - \frac{x}{2}u^2d - \frac{xg}{2}(2sC + D) = (y_1 - xs)gC. \tag{5.1.3}$$

If $x = 0$, the equation (5.1.3) is true since $C = \frac{z+duy_2}{gy_1}$. If $x \neq 0$, the equation (5.1.3) is still true by using equation (2.5.6).

Let us prove the equation (5.1.2).

$$\begin{aligned}
\text{LHS of (5.1.2)} &= (sH + u\gamma).(CH + u\gamma) - \frac{g}{2}(t^2 + s^2 - \frac{d}{g}u^2) \\
&= sCg - u^2d - \frac{g}{2}(2sC + D - \frac{d}{g}u^2) = \text{RHS of (5.1.2)}
\end{aligned}$$

□

Definition 5.1.2. Fix a choice of Maciocia's coordinate (H, γ, u). We define the Mukai vector

$$\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}, \mathrm{ch}') := \left(1, CH + u\gamma - \frac{3}{4}\mathrm{K}_S, -\frac{3}{4}\mathrm{K}_S.[CH + u\gamma] - \frac{1}{2}\chi(\mathcal{O}_S) + \frac{11}{32}\mathrm{K}_S^2\right),$$

where the center $C = C(\mathrm{ch}, \mathrm{ch}')$. If $\mathrm{ch}_0 = 0$ and $\mathrm{ch}_1.H > 0$, then the center is given by $C = \frac{z+duy_2}{gy_1}$ (Theorem 2.5.2), which is independent of ch'. In this case, the vector $\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}, \mathrm{ch}')$ is also independent of ch', and we denote it by

$$\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}) = \mathbf{t}_{(H,\gamma,u)}(0, \mathrm{ch}_1, \mathrm{ch}_2) := \mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}, \mathrm{ch}').$$

Lemma 5.1.3. *If $\Im Z_{\omega,\beta}(\mathrm{ch}) > 0$, then*

$$w_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=} \left(\frac{g}{2}D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2}u^2\right)(0, 0, -1) + \mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}, \mathrm{ch}'). \tag{5.1.4}$$

Proof. This is a direct computation by using equations (3.2.3, 5.1.1, 5.1.2). □

5.2 Local Bayer-Macrì decomposition

Similar as Section 4.2, we decompose w_σ in three cases, according to the the dimension of support of objects with invariants ch. *Assume* there is a flat family $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(M_\sigma(\mathrm{ch}) \times S)$ and denote the Mukai morphism by $\theta_{\sigma,\mathcal{E}}$.

5.2.1 Supported in dimension 0

Fix $\mathrm{ch} = (0, 0, n)$, with n a positive integer. Fix a Maciocia's coordinate (H, γ, u). Since $t > 0$ is the trivial chamber and there is no wall on $\Pi_{(H,\gamma,u)}$, we obtain $w_\sigma \overset{\mathbb{R}_+}{=\!=} \big(0, H, (\beta - \frac{3}{4}\mathrm{K}_S)H\big)$, and the nef line bundle $\ell_\sigma = \theta_{\sigma,\mathcal{E}}(0, H, sH^2 - \frac{3}{4}KH)$ on the moduli space $M_\sigma(\mathrm{ch}) \cong \mathrm{Sym}^n(S)$, which is independent of s.

5.2.2 Supported in dimension 1

Fix a Maciocia's coordinate (H, γ, u). We assume that $\mathrm{ch} = (0, \mathrm{ch}_1, \mathrm{ch}_2)$ with $\mathrm{ch}_1.H > 0$. Denote

$$\mathcal{S} := \theta_{\sigma,\mathcal{E}}(0, 0, -1), \quad \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}) := \theta_{\sigma,\mathcal{E}}(\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch})).$$

By Lemma 3.1.4, $\mathcal{S} = \lambda_{\mathcal{E}}(0, 0, 1)$.

Theorem 5.2.1. *(Local Bayer-Macrì decomposition for* $\mathrm{ch}_0 = 0$ *and* $\mathrm{ch}_1.H > 0$.*) Assume* $\mathrm{ch} = (0, \mathrm{ch}_1, \mathrm{ch}_2)$ *with* $\mathrm{ch}_1.H > 0$ *for a choice of Maciocia's coordinate* (H, γ, u).

(a). $w_{\sigma \in W(\mathrm{ch},\mathrm{ch}')}$ *has a decomposition*

$$w_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=} \left(\frac{g}{2} D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2} u^2\right)(0, 0, -1) + \mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}), \qquad (5.2.1)$$

where $(0, 0, -1)$, $\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}) \in \mathbf{v}^\perp$. *Moreover* $r = \mathrm{ch}'_0 \neq 0$ *and the coefficient before* $(0, 0, -1)$ *is expressed in terms of potential destabilizing Chern characters* $\mathrm{ch}' = (r, c_1 H + c_2 \gamma + \delta', \chi)$*:*

$$\frac{g}{2} D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2} u^2 = \frac{\chi - gCc_1 + udc_2}{r}. \qquad (5.2.2)$$

(b). Assume that there is a flat family $\mathcal{E} \in \mathrm{D^b}(M_\sigma(\mathrm{ch}) \times S)$. Then the Bayer-Macrì nef line bundle on the moduli space $M_\sigma(\mathrm{ch})$ has a decomposition

$$\ell_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} = \left(\frac{g}{2}D(\mathrm{ch},\mathrm{ch}') + \frac{d}{2}u^2\right)\mathcal{S} + \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}). \tag{5.2.3}$$

Proof. Parts (a) follows from computation. The Mukai vector $\mathbf{v}$ is given by

$$\mathbf{v} = (0, \mathrm{ch}_1, \mathrm{ch}_2 - \frac{1}{4}\mathrm{ch}_1.\mathrm{K}_S).$$

So $(0, 0, -1) \in \mathbf{v}^\perp$ by the definition equation (3.1.2) and the formula (2.3.6). To show $\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}) \in \mathbf{v}^\perp$, we can either directly compute the Mukai bilinear form:

$$\langle \mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}), \mathbf{v}\rangle_S = (CH + u\gamma - \frac{3}{4}\mathrm{K}_S)\cdot \mathrm{ch}_1 - (\mathrm{ch}_2 - \frac{1}{4}\mathrm{ch}_1.\mathrm{K}_S - \frac{1}{2}\mathrm{ch}_1.\mathrm{K}_S) = 0,$$

or notice the relation (5.1.4) and the fact $w_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} \in \mathbf{v}^\perp$, $(0, 0, -1) \in \mathbf{v}^\perp$. Recall the equation (2.5.5) for $D(\mathrm{ch}, \mathrm{ch}')$. Since $x = \mathrm{ch}_0 = 0$, we obtain $r \neq 0$. The relation (5.2.2) is then derived by using equation (2.5.7). Part (b) follows from part (a) by applying the Mukai morphism $\theta_{\sigma,\mathcal{E}}$. □

5.2.3 Supported in dimension 2

Assume that $\mathrm{ch}_0 \neq 0$. We define the following special kinds of Mukai vectors with respective to ch. In particular, the definition of Mukai vector $\mathbf{m}(H, \mathrm{ch})$ is motivated by equation (4.4.3).

Definition 5.2.2. Fix Chern characters $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ with $\mathrm{ch}_0 \neq 0$. Define

$$\begin{aligned}
\mathbf{w}(\mathrm{ch}) &:= \left(1, -\frac{3}{4}\mathrm{K}_S, -\frac{\mathrm{ch}_2}{\mathrm{ch}_0} - \frac{1}{2}\chi(\mathcal{O}_S) + \frac{11}{32}\mathrm{K}_S^2\right), \\
\mathbf{m}(H, \mathrm{ch}) &:= \left(0, H, (\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \frac{3}{4}\mathrm{K}_S).H\right), \text{ where } H \in N^1(S),
\end{aligned}$$

$$\mathbf{u}(\mathrm{ch}) := \mathbf{w}(\mathrm{ch}) + \mathbf{m}(\frac{1}{2}\mathrm{K}_S, \mathrm{ch}) = \left(1, -\frac{1}{4}\mathrm{K}_S, -\frac{\mathrm{ch}_2}{\mathrm{ch}_0} + \frac{\mathrm{ch}_1.\mathrm{K}_S}{2\mathrm{ch}_0} - \frac{1}{2}\chi(\mathcal{O}_S) - \frac{1}{32}\mathrm{K}_S^2\right).$$

Lemma 5.2.3. *We have the following three perpendicular relations for Mukai vectors:*

$$\mathbf{m}(H, \mathrm{ch}),\ \mathbf{w}(\mathrm{ch}),\ \mathbf{u}(\mathrm{ch}) \in \mathbf{v}^\perp.$$

Proof. The perpendicular relations can be checked directly by definition equation (3.1.2) and equations (2.3.5, 2.3.6). □

Theorem 5.2.4. *(Local Bayer-Macrì decomposition for objects supported in dimension* 2.*)*

(a). If $\mathrm{ch}_0 \neq 0$ *and* $\Im Z(\mathrm{ch}) > 0$*, then there is a decomposition (up to a positive scalar):*

$$\begin{aligned} w_{\omega,\beta}(\mathrm{ch}) &\overset{\mathbb{R}_+}{=\!=} \mu_\sigma(\mathrm{ch})\mathbf{m}(\omega,\mathrm{ch}) + \mathbf{m}(\beta,\mathrm{ch}) + \mathbf{w}(\mathrm{ch}) && (5.2.4)\\ &= \mu_\sigma(\mathrm{ch})\mathbf{m}(\omega,\mathrm{ch}) + \mathbf{m}(\alpha,\mathrm{ch}) + \mathbf{u}(\mathrm{ch}), && (5.2.5)\end{aligned}$$

where $\mathbf{m}(\omega,\mathrm{ch})$, $\mathbf{m}(\beta,\mathrm{ch})$, $\mathbf{m}(\alpha,\mathrm{ch})$, $\mathbf{w}(\mathrm{ch})$, $\mathbf{u}(\mathrm{ch}) \in \mathbf{v}^\perp$.

(b). Assume there is a flat family $\mathcal{E}$*. Then the Bayer-Macrì line bundle class* $\ell_{\sigma_{\omega,\beta}}$ *has a decomposition in* $N^1(M_\sigma(\mathrm{ch}))$*:*

$$\ell_{\sigma_{\omega,\beta}} = \mu_\sigma(\mathrm{ch})\theta_{\sigma,\mathcal{E}}(\mathbf{m}(\omega,\mathrm{ch})) + \theta_{\sigma,\mathcal{E}}(\mathbf{m}(\beta,\mathrm{ch})) + \theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch})). \qquad (5.2.6)$$

Proof. Recall the equation (3.2.3). To show the relation (5.2.4), we only need to check that

$$\frac{\mathrm{ch}_1}{\mathrm{ch}_0}.[\mu_\sigma(\mathrm{ch})\omega + \beta] - \frac{\mathrm{ch}_2}{\mathrm{ch}_0} = \beta.[\mu_\sigma(\mathrm{ch})\omega + \beta] - \frac{1}{2}(\omega^2 + \beta^2). \qquad (5.2.7)$$

Recall that the Bridgeland slope is given by $\mu_\sigma(\mathrm{ch}) = \frac{-\Re Z(\mathrm{ch})}{\Im Z(\mathrm{ch})}$. Then by equation (2.1.1), we have

$$\mu_\sigma(\mathrm{ch}) = \frac{\mathrm{ch}_2 - \frac{1}{2}\mathrm{ch}_0\,(\omega^2 - \beta^2) - \mathrm{ch}_1.\beta}{\omega.\,(\mathrm{ch}_1 - \mathrm{ch}_0\beta)}.$$

So

$$\mu_\sigma(\mathrm{ch})\omega.\left(\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \beta\right) = \frac{\mathrm{ch}_2}{\mathrm{ch}_0} - \frac{1}{2}\left(\omega^2 - \beta^2\right) - \frac{\mathrm{ch}_1}{\mathrm{ch}_0}.\beta.$$

Therefore we have the relation (5.2.7). The equation (5.2.5) follows from equation (5.2.4) and the relation $\alpha = \beta - \frac{1}{2}\mathrm{K}_S$. Part (b) follows directly by applying the Mukai morphism $\theta_{\sigma,\mathcal{E}}$. □

Remark 5.2.5. The condition $\mathrm{ch}_0 \neq 0$ and $\Im Z(\mathrm{ch}) = 0$ means exactly that $\sigma \in \mathtt{UW}$ or $\sigma \in \mathtt{DUW}$ by Remark 2.6.3 and Definition 4.3.1. Since the $\mathtt{UW}$ cannot be touched from the Gieseker chamber, we consider the case $\sigma \in \mathtt{DUW}$. The easy computation shows that $\sigma \in \mathtt{DUW}$ if and only if $\mu_\sigma(\mathrm{ch}) = +\infty$. So the equation (4.4.3) is the limit of equation (5.2.4).

Recall Lemma 3.3.1 and Corollary 2.6.2. We obtain

$$\begin{aligned} w_{\omega,-\beta}(-\mathrm{ch}^*) &\overset{\mathbb{R}_+}{=\!=} \mu_{\Phi(\sigma)}(-\mathrm{ch}^*)\mathbf{m}(\omega,-\mathrm{ch}^*)+\mathbf{m}(-\beta,\mathrm{ch})+\mathbf{w}(-\mathrm{ch}^*), \\ \ell_\sigma \cong \ell_{\Phi(\sigma)} &= \mu_{\Phi(\sigma)}(-\mathrm{ch}^*)\theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{m}(\omega,-\mathrm{ch}^*)) \\ &\quad +\theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{m}(-\beta,-\mathrm{ch}^*))+\theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{w}(-\mathrm{ch}^*)). \end{aligned}$$

Then $\theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{w}(-\mathrm{ch}^*)) \cong \theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch}))$. Since $\mu_{\Phi(\sigma)}(-\mathrm{ch}^*) = -\mu_\sigma(\mathrm{ch})$, we get $\theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{m}(\omega,-\mathrm{ch}^*)) \cong -\theta_{\sigma,\mathcal{E}}(\mathbf{m}(\omega,\mathrm{ch}))$.

Notation 5.2.6. Assume $\mathrm{ch}_0 > 0$. Denote

$$\widetilde{L} := \theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{m}(L,-\mathrm{ch}^*)) \cong -\theta_{\sigma,\mathcal{E}}(\mathbf{m}(L,\mathrm{ch}))$$

for a line bundle L on S. Denote $\mathcal{B}_0 := -\theta_{\sigma,\mathcal{E}}(\mathbf{u}(\mathrm{ch}))$. Then

$$\theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch})) = \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0. \tag{5.2.8}$$

Recall $\alpha = \beta - \frac{1}{2}\mathrm{K}_S$. Denote

$$\mathcal{B}_\alpha := \widetilde{\beta} - \theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{w}(-\mathrm{ch}^*)) \cong \widetilde{\beta} - \theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch})) = \widetilde{\alpha} + \mathcal{B}_0. \tag{5.2.9}$$

$\widetilde{L}$, $\mathcal{B}_\alpha$ and $\mathcal{B}_0$ are line bundles on $M_\sigma(\mathrm{ch})$.

5.3 Le Potier's construction of determinant line bundles

Assume $\mathrm{ch}_0 > 0$. Assume that $\sigma \in \mathtt{GC}$ with respect to a choice of Maciocia's coordinate (H,γ,u). Then $\widetilde{H}$ and $\mathcal{B}_0$ are line bundles on $M_H(\mathrm{ch})$. Line bundles $\widetilde{H}$ and $\mathcal{B}_0$ have direct relations with the determinant line bundles constructed by Le Potier. Let us first recall Le Potier's construction [LeP92] of two particular line bundles $\mathcal{L}_0$ and $\mathcal{L}_1$ on the moduli space $M_H(\mathrm{ch})$ via the Donaldson morphism. The line bundles $\mathcal{L}_i$ play an important role in the geometry of moduli spaces (e.g. [HL10, Chapter 8]).

Definition 5.3.1. [LeP92, Section 3] [HL10, Definition 8.1.9] Let $H \in \mathrm{Amp}(S)$ be an ample divisor and let $h := [\mathcal{O}_H]$ be its class in $K(S)$. Let $s \in S$ be a closed point. Let $u_i(\mathrm{ch}) = -\mathrm{ch}_0 \cdot h^i + \chi(\mathrm{ch} \cdot h^i) \cdot [\mathcal{O}_s]$ and let $\mathcal{L}_i \in M_H(\mathrm{ch})$ be the line bundle $\mathcal{L}_i := \lambda_{\mathcal{E}}(u_i(\mathrm{ch}))$ for $i = 0, 1$.

Denote $\mathbf{a}$ as Chern characters of $u_0(\mathrm{ch})$ and denote $\mathbf{b}$ as Chern characters of $u_1(\mathrm{ch})$. Then

$$\begin{aligned}
\mathbf{a} &= (-\mathrm{ch}_0, 0, \mathrm{ch}_2 - \frac{1}{2}\mathrm{ch}_1.\mathrm{K}_S + \mathrm{ch}_0\chi(\mathcal{O}_S)),\\
\mathbf{b} &= (0, -\mathrm{ch}_0 H, (\mathrm{ch}_1 - \frac{1}{2}\mathrm{ch}_0\mathrm{K}_S).H).
\end{aligned}$$

Direct computation shows that $u_i(\mathrm{ch}) \in \mathrm{ch}^\sharp$. So we can apply the Donaldson morphism $\lambda_{\mathcal{E}}$ and above $\mathcal{L}_i$ are well-defined. Recall the relation between Donaldson morphism and Mukai morphism in Lemma 3.1.4, we have

$$\mathcal{L}_0 = \lambda_{\mathcal{E}}(u_0(\mathrm{ch})) = -\theta_{\mathsf{C},\mathcal{E}}(v(\mathbf{a}^*)), \quad \mathcal{L}_1 = \lambda_{\mathcal{E}}(u_1(\mathrm{ch})) = -\theta_{\mathsf{C},\mathcal{E}}(v(\mathbf{b}^*)). \tag{5.3.1}$$

We compute $v(\mathbf{a}^*)$ by using equations (2.3.1, 2.3.5):

$$\begin{aligned}
v(\mathbf{a}^*) &= v(-\mathrm{ch}_0, 0, \mathrm{ch}_2 - \frac{1}{2}\mathrm{ch}_1.\mathrm{K}_S + \mathrm{ch}_0\chi(\mathcal{O}_S))\\
&= (-\mathrm{ch}_0, -\frac{1}{4}(-\mathrm{ch}_0))\mathrm{K}_S, \mathrm{ch}_2 - \frac{1}{2}\mathrm{ch}_1.\mathrm{K}_S + \mathrm{ch}_0\chi(\mathcal{O}_S)\\
&\qquad + \frac{1}{2}(-\mathrm{ch}_0)\left(\chi(\mathcal{O}_S) - \frac{1}{16}\mathrm{K}_S^2\right))\\
&= -\mathrm{ch}_0\left(1, -\frac{1}{4}\mathrm{K}_S, -\frac{\mathrm{ch}_2}{\mathrm{ch}_0} + \frac{\mathrm{ch}_1.\mathrm{K}_S}{2\mathrm{ch}_0} - \frac{1}{2}\chi(\mathcal{O}_S) - \frac{1}{32}\mathrm{K}_S^2\right)\\
&= -\mathrm{ch}_0\mathbf{u}(\mathrm{ch}).
\end{aligned} \tag{5.3.2}$$

Similarly, we have

$$\begin{aligned}
v(\mathbf{b}^*) &= v(0, \mathrm{ch}_0 H, (\mathrm{ch}_1 - \frac{1}{2}\mathrm{ch}_0\mathrm{K}_S).H)\\
&= (0, \mathrm{ch}_0 H, (\mathrm{ch}_1 - \frac{1}{2}\mathrm{ch}_0\mathrm{K}_S).H - \frac{1}{4}(\mathrm{ch}_0 H).\mathrm{K}_S)\\
&= \mathrm{ch}_0\mathbf{m}(H, \mathrm{ch}).
\end{aligned} \tag{5.3.3}$$

Proposition 5.3.2. *Assume that* ch *satisfies the condition* (C) *as in Assumption 4.2.3. Choose $H \in \mathrm{Amp}(S)$ and let $\mathcal{E}$ be a flat family over $M_H(\mathrm{ch})$. Then*

$$\mathcal{L}_0 = -\mathrm{ch}_0\mathcal{B}_0, \quad \mathcal{L}_1 = \mathrm{ch}_0\widetilde{H}. \tag{5.3.4}$$

Proof. By equations (5.3.1, 5.3.2, 5.3.3), we have

$$\begin{aligned}
\mathcal{L}_0 &= -\theta_{\mathsf{GC},\mathcal{E}}(v(\mathbf{a}^*)) = \theta_{\mathsf{GC},\mathcal{E}}(\mathrm{ch}_0\mathbf{u}(\mathrm{ch})) = -\mathrm{ch}_0\mathcal{B}_0,\\
\mathcal{L}_1 &= -\theta_{\mathsf{GC},\mathcal{E}}(v(\mathbf{b}^*)) = -\theta_{\mathsf{GC},\mathcal{E}}(\mathrm{ch}_0\mathbf{m}(H, \mathrm{ch})) = \mathrm{ch}_0\widetilde{H}.
\end{aligned}$$

□

We recall one basic property of $\mathcal{L}_0$. So it is also a property of $\mathcal{B}_0$. We first give some notations. Let S be a smooth projective surface. Fix an ample line bundle H on S. Let ch be the Chern characters of a torsion free sheaf E. Denote $\mathrm{ch}(m) := \mathrm{ch}(E \otimes H^{\otimes m})$.

Proposition 5.3.3. *[HL10, Theorem 8.1.11] With notations above, for $m \gg 0$ the line bundle $\mathcal{L}_0 = \lambda(u_0(\mathrm{ch}(m)))$ on $M_H(\mathrm{ch}(m))$ is relatively ample with respect to the determinant morphism $\det : M_H(\mathrm{ch}(m)) \to \mathrm{Pic}(S)$.*

Problem 5.3.4. *For the case $\mathrm{ch}_0 = 0$ and $\mathrm{ch}_1.H > 0$ in Section 5.2.2, we observe that $\mathcal{S}$ is an analogue of $\mathcal{L}_1$ and $\mathcal{T}_{(H,\gamma,u)}$ is an analogue of $\mathcal{L}_0$ (or $\mathcal{B}_0$). Above Proposition gives the geometric property of $\mathcal{B}_0$. Is there some similar geometric property of $\mathcal{S}$?*

5.4 Global Bayer-Macrì decomposition

A priori the line bundles $\mathcal{S}$, $\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch})$ in dimension 1 case, or $\widetilde{L}$, $\mathcal{B}_\alpha$ and $\mathcal{B}_0$ in dimension 2 case, are only on their own moduli space $M_\sigma(\mathrm{ch})$. *If* we can identify the Néron-Severi groups as in Section 3.4, then we regard them as line bundles on any birational models of $M_\sigma(\mathrm{ch})$. The divisorial contraction of moduli space $M_\sigma(\mathrm{ch}) \dashrightarrow M'$ is a limit case. Namely, $\mathcal{S}$ or $\mathcal{B}_0$ may disappear as divisors on M'.

5.4.1 Supported in dimension 1

Theorem 5.4.1. *Assume $\mathrm{ch} = (0, \mathrm{ch}_1, \mathrm{ch}_2)$ with $\mathrm{ch}_1.H > 0$ for a choice of Maciocia's coordinate (H, γ, u). Assume that we can identify different Néron-Severi groups as Section 3.4.*

(a). $\mathcal{S}$ and $\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch})$ have global meaning and equation (5.2.3) is the correspondence from Bridgeland walls to Mori walls.

(b). If $\sigma_{\omega,\beta}$ is a generic stability condition (i.e. not in a wall for ch), then the image of the Bayer-Macrì map in $\mathrm{NS}(M_{\sigma_{\omega,\beta}}(\mathrm{ch}))$ is of rank one more than the Picard number of the surface.

Proof. Let us prove part (a). If we can identify different Néron-Severi groups as Section 3.4, we then fix a base stability condition $\sigma \in \mathtt{SC}$ and obtain

$$\mathcal{S} = \theta_{\mathtt{SC},\mathcal{E}}((0,0,-1)) \in N^1(M_{(\alpha,\omega)}(\mathrm{ch})),$$

$$\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}) = \theta_{\mathtt{SC},\mathcal{E}}(\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch})) \in N^1(M_{(\alpha,\omega)}(\mathrm{ch})).$$

For the part (b), since $\sigma_{\omega,\beta}$ is in a chamber for ch, we can perturb ω in the ample cone $\mathrm{Amp}(S)$ and still get a generic stability condition in the same chamber. So the image of the Bayer-Macrì map in $\mathrm{NS}(M_{\sigma_{\omega,\beta}}(\mathrm{ch}))$ is of rank one more than the Picard number of the surface, and this extra one divisor is given by $\mathcal{S}$. $\square$

Recall that the line bundle $\mathcal{S}$ is conjectured to induce the support morphism (Conjecture 4.6.3). Let $s \in S$ be a smooth point. Denote $M = M_{(\alpha,\omega)}(\mathrm{ch})$. we have [HL10, Example 8.1.3]

$$\mathcal{S} = \lambda_{\mathcal{E}}(0,0,1) = (p_M)_*(\det(\mathcal{E})|_{M\times\{s\}}).$$

We extend the equations (5.2.2, 2.5.7) to the case $r = 0$. Then $D = R^2 - C^2 = +\infty$ since the center C is fixed and the radius R is $+\infty$. The wall is nothing but the Simpson wall $\mathtt{SW}$: $t = +\infty$. So the equation (4.4.4) is the limit case of equation (5.2.1), and the equation (4.6.3) is the limit case of equation (5.2.3).

The coefficient $\frac{\chi - gCc_1 + udc_2}{r}$ in equation (5.2.3) is important for computing nef cone and effective cone of the moduli space $M_\sigma(\mathrm{ch})$. We are interested in the geometric property of $\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch})$ (Problem 5.3.4).

Example 5.4.2. We continue Example 4.6.1 [Woo13]. Since the Picard group of $\mathbb{P}^2$ is generated by H, we have $\gamma = 0$ and $d = u = 0$. Denote the potential destabilizing Chern characters by $\mathrm{ch}' = (r, c_1 H, \mathrm{ch}_2')$. The potential walls are semicircles with fixed center $(C, 0)$ in the half-plane $\Pi_{(H,0,0)}$, where $C = \frac{\chi}{\mu} - \frac{3}{2}$. The equation (5.2.3) becomes

$$\ell_\sigma = \frac{\mathrm{ch}_2' - c_1 C}{r}\mathcal{S} + \mathcal{T}_{(H,0,0)}(\mathrm{ch}), \tag{5.4.1}$$

where $\mathcal{T}_{(H,0,0)}(\mathrm{ch}) = \theta_{\sigma,\mathcal{E}}\left(1, (C + \frac{9}{4})H, \frac{9}{4}C + \frac{83}{32}\right)$. Woolf showed that $N(\mu,\chi)$ is a Mori dream space and of Picard rank 2. So by Example 3.4.3, we obtain the global meaning of $\mathcal{S}$ and $\mathcal{T}_{(H,0,0)}$, and regard them as the generator of the Néron-Severi group. One boundary of $\mathrm{Nef}(N(\mu,\chi))$ is given by $\mathcal{S}$ where $\sigma \in \mathtt{SW}$.

The other boundary of $\mathrm{Nef}(N(\mu,\chi))$ is given by ℓ_σ where σ is in the biggest semicircle (but not the $\texttt{SW}$), and the coefficient $\frac{\mathrm{ch}_2'-c_1C}{r}$ obtains the maximal value for variables $\mathrm{ch}'=(r,c_1H,\mathrm{ch}_2')$. The maximal value is calculated by Woolf [Woo13, Corollary 7.6].

Example 5.4.3. (Arcara and Bertram [AB13]). Let S be a K3 surface with ample divisor H and $\mathrm{Pic}(S)=\mathbb{Z}H$. Fix $\mathrm{ch}=(0,H,\frac{H^2}{2})$. Then the center is $(C,0)$ in the (s,t)-half-plane $\Pi_{(H,0,0)}$, where $C=\frac{1}{2}$. They further fixed $s=\frac{1}{2}$ and studied mini-walls, i.e. the intersection of semicircles with this line. If $s=\frac{1}{2}$ and $t>\frac{1}{6}$, the destabilized sequence is

$$0\to(I_W)^\vee[1]\to E\to I_Z(H)\to 0,$$

where $W,Z\subset S$ are 0-dimensional subschemes of the same length with relation $|W|=|Z|<\frac{H^2}{8}$ [AB13, Theorem 3.7]. Let $\mathrm{ch}'=\mathrm{ch}((I_W)^\vee[1])=(-1,0,|W|)$. By equation (2.5.7), we get $D(\mathrm{ch},\mathrm{ch}')=-\frac{2|W|}{H^2}$, and $R=\sqrt{C^2+D}=\sqrt{\frac{1}{4}-\frac{2|W|}{H^2}}$. The first non-trivial wall is given by the radius $\frac{1}{2}$, with $|W|=0$, i.e. the destabilizing object $\mathcal{O}_S[1]$. The line bundle in equation (5.2.3) becomes

$$\ell_\sigma=-|W|\mathcal{S}+\mathcal{T}_{(H,0,0)}(\mathrm{ch})\text{ with }0\le|W|<\frac{H^2}{8},$$

where $\mathcal{T}_{(H,0,0)}(\mathrm{ch})=\theta_{\sigma,\mathcal{E}}(1,\frac{1}{2}H,-1)$.

5.4.2 Supported in dimension 2

By applying the derived dual functor if necessary, we can further assume $\mathrm{ch}_0>0$ in Theorem 5.2.4. The geometric meaning of the decomposition in Theorem 5.2.4 is given as follows.

Theorem 5.4.4. *Fix* ch *with* $\mathrm{ch}_0>0$. *Assume that (i) there is a flat family* $\mathcal{E}$*; (ii)* $M_{(\alpha,\omega)}(\mathrm{ch})$ *is irreducible; (iii)* $\partial M_{(\alpha,\omega)}(\mathrm{ch})\neq\emptyset$*; (iv)* $U_\omega(\mathrm{ch})$ *is of positive dimension; (v) we can identify the Néron-Severi groups as Section 3.4. Then the following conclusions hold.*

(a). There is a Bayer-Macrì decomposition for the line bundle $\ell_{\sigma_{\omega,\beta}}$*:*

$$\ell_{\sigma_{\omega,\beta}}\cong(-\mu_{\sigma_{\omega,\beta}}(\mathrm{ch}))\widetilde{\omega}-\mathcal{B}_\alpha=(-\mu_{\sigma_{\omega,\beta}}(\mathrm{ch})\widetilde{\omega}-\widetilde{\alpha}-\mathcal{B}_0. \tag{5.4.2}$$

(b). The line bundle $\widetilde{\omega}$ induces the GU morphism from the (α,ω)-Gieseker semistable moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ to the Uhlenbeck space $U_\omega(\mathrm{ch})$.

(c). If $\sigma_{\omega,\beta}$ is a generic stability condition (i.e. not in a wall for ch*), then the image of the Bayer-Macrì map in $\mathrm{NS}(M_{\sigma_{\omega,\beta}}(\mathrm{ch}))$ is of rank one more than the Picard number of the surface.*

(d). If $\mathrm{ch}_0 = 2$, the divisor $\mathcal{B}_\alpha$ is the α-twisted boundary divisor of the induced GU morphism. In particular, in the case of $\alpha = 0$, the divisor $\mathcal{B}_0 = -\theta_{\sigma,\mathcal{E}}(\mathbf{u}(\mathrm{ch}))$ is the (untwisted) boundary divisor from the ω-semistable Gieseker moduli space $M_\omega(\mathrm{ch})$ to the Uhlenbeck space $U_\omega(\mathrm{ch})$.

Proof. If we can identify the Néron-Severi groups as Section 3.4, then $\widetilde{\omega}$ and $\mathcal{B}_\alpha$ have global meaning, and equation (5.2.6) implies Part (a). Part (b) follows from Proposition 4.5.1. In particular, the line bundle $\widetilde{\omega}$ induces the GU morphism. If $\sigma_{\omega,\beta}$ is in a chamber for ch, we can perturb ω in the ample cone $\mathrm{Amp}(S)$ and still get a generic stability condition in the same chamber. So the image of the Bayer-Macrì map in $\mathrm{NS}(M_{\sigma_{\omega,\beta}}(\mathrm{ch}))$ is of rank one more than the Picard number of the surface. This shows Part (c). If $\mathrm{ch}_0 = 2$ and $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$, the GU c is a divisorial contraction by Lemma 4.5.3, and $\mathcal{B}_\alpha$ is the boundary divisor from the α-twisted moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ to the Uhlenbeck space $U_\omega(\mathrm{ch})$. In particular, the $\mathcal{B}_0$ is the untwisted boundary divisor. This shows Part (d). $\square$

Remark 5.4.5. If ch satisfies the condition (C) as in Assumption 4.2.3, then the universal family $\mathcal{E}$ exists [HL10, Corollary 4.6.7]. Moreover, if $\mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2$ is large enough in the sense of [HL10, Theorem 9.4.3], then the moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ is irreducible.

Remark 5.4.6. *Can every line bundle on moduli space be written as determinant line bundle?* The author learnt this interesting question from Bayer at UIC workshop on Bridgeland stability conditions (2013). The question is asking the surjectivity of the local Bayer-Macrì map, and we already know the image of the local Bayer-Macrì map, which partially answers the question. Denote $\rho(X) = \mathrm{rank}(\mathrm{NS}(X))$. If $\rho(M_{\sigma_{\omega,\beta}}(\mathrm{ch})) = \rho(S) + 1$, then every line bundle on the moduli space can be generated by the image of a Bayer-Macrì map. In particular this is true in the case of $\mathrm{ch} = (2, \mathrm{ch}_1, \mathrm{ch}_2)$ with $\mathrm{ch}_2 \ll 0$ by Li [Li96], and in the case of Hilbert scheme of n-points by the classical work of Fogarty [Fog68].

The equation (5.4.2) gives a direct link of the ample cone of the surface to the ample cone of the moduli space, which links the geometry of surface to the geometry of the moduli space. Let ω run over the ample cone of S. Then the image $\ell_{\omega,\beta}$ gives partial of ample cone of the moduli space. It is interesting to ask if the full ample cone of the moduli space can be obtained in this way by knowing the surjectivity of the local Bayer-Macrì map.

Example 5.4.7. If $\mathrm{ch} = (1, 0, -n)$, then the Gieseker-Uhlenbeck morphism is the Hilbert-Chow morphism $h : S^{[n]} \to S^{(n)}$, which maps the Hilbert scheme of n-points on S to the symmetric product $S^{(n)}$. In particular,

$$\widetilde{H} = \mathcal{L}_1 = h^*(\mathcal{O}_{S^{(n)}}(1)),$$

which induces the Hilbert-Chow morphism [HL10, Example 8.2.9]. The boundary divisor of Hilbert-Chow morphism is $\mathrm{B} := \mathrm{B}_n := \{\xi \in S^{[n]} : |\mathrm{Supp}(\xi)| < n\}$. It is known from the Appendix in [BSG91] that $\frac{1}{2}\mathrm{B}$ is an integral divisor and $\mathcal{B}_0 = -\mathcal{L}_0 = \frac{1}{2}\mathrm{B}$.

Chapter 6

Bridgeland-Mori correspondence for geometric Bridgeland stability

Let $M := M_{(\alpha,\omega)}$ be the α-twisted ω-semistable Gieseker moduli space. We assume that M is a Mori dream space (Section C.2). There is a Mori chamber decomposition of its pseudo-effective cone $\overline{\mathrm{Eff}}(M)$ into rational polyhedra (Theorem C.2.4). For each divisor D in the interior of polytope, there is a rational map $\phi_D : M \dashrightarrow M(D)$, where $M(D) := \mathrm{Proj}(R(M, D))$ and $R(M, D) := \bigoplus_{m\geq 0} H^0(M, mD)$ is the section ring. On the other hand, Bridgeland conjectured that changing Bridgeland stability conditions also produce birational models of M. The two aspect of birational geometry of M are tightly linked by the observation/speculation of Arcara, Bertram, Coskun, Huizenga [ABCH13] in the case of $\mathbb{P}^2$, as well as the work of Bayer and Macrì [BM14a, BM14b] in the case of K3. We give the following correspondence by assuming the existence of the universal family and the identification of Néron-Severi groups.

Recall we have two examples of such identifications. One example is Bridgeland moduli spaces over K3 surface with primitive Mukai vector (Example 3.4.1). The other example is the Mori dream spaces (Example 3.4.3). There is a Mori chamber decomposition for each case.

6.1 Bridgeland-Mori correspondence for geometric Bridgeland stability conditions

Theorem 6.1.1. *(The Bridgeland-Mori correspondence for objects supported in dimension 2 and geometric Bridgeland stability conditions.) Fix a Maciocia's coordinate* (H, γ, u)*. Assume that (i)* $\mathrm{ch}_0 > 0$*; (ii) there is a flat family* $\mathcal{E}$*; (iii) we can identify the Néron-Severi groups. Then there is a correspondence from the Bridgeland wall* $W(\mathrm{ch}, \mathrm{ch}')$ *on the half-plane* $\Pi_{(H,\gamma,u)}$ *with center* $C(\mathrm{ch}, \mathrm{ch}')$ *to the nef line bundle on the moduli space* $M_\sigma(\mathrm{ch})$*:*

$$\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} = -C(\mathrm{ch}, \mathrm{ch}')\widetilde{H} - u\widetilde{\gamma} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0. \tag{6.1.1}$$

Proof. With the existence of the flat family $\mathcal{E}$, we have the Mukai morphism $\theta_{\sigma,\mathcal{E}}$. With the identification of Néron-Severi groups, the line bundles $\widetilde{H}$, $\widetilde{\gamma}$, $\widetilde{\mathrm{K}_S}$ and $\mathcal{B}_0$ have global meaning. The equation (6.1.1) follows from a direct computation:

$$\begin{aligned} \ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} &= \theta_{\sigma,\mathcal{E}}(\mathbf{m}(\mu_\sigma(\mathrm{ch})\omega + \beta, \mathrm{ch})) + \theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch})) \quad \text{by equation (5.2.6)} \\ &= -C(\mathrm{ch}, \mathrm{ch}')\widetilde{H} - u\widetilde{\gamma} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0. \quad \text{by equations (5.1.1, 5.2.8)} \end{aligned}$$

□

Remark 6.1.2. The problem about the projectivity of $M_\sigma(\mathrm{ch})$ and the problem about the finitely generation property of the section ring of ℓ_σ, both of them are unknown in general, are related.

Remark 6.1.3. The above correspondence is only for numerical geometric stability conditions $\sigma_{\omega,\beta} \in \mathrm{Stab}^\dagger(S)$ (Remark 2.4.2). Toda [Tod13, Tod14] constructed new Bridgeland stability conditions which are not geometric, by using the tilting of perverse coherent sheaves. He studied the simplest invariants $\mathrm{ch} = (0, 0, 1)$, yet obtained quite interesting birational geometry of surfaces. It is interesting to study birational geometry of Bridgeland moduli spaces for stability conditions beyond the geometric stability conditions.

Corollary 6.1.4. *(The Bridgeland-Mori correspondence for Mori dream space) Assume that* $M := M_{(\alpha,\omega)}$ *is a Mori dream space. Then there is a correspondence from the Bridgeland wall* $W(\mathrm{ch}, \mathrm{ch}')$ *on the half-plane* $\Pi_{(H,\gamma,u)}$ *with center* $C(\mathrm{ch}, \mathrm{ch}')$ *to the nef line bundle on the moduli space* $M_\sigma(\mathrm{ch})$*:*

$$\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} = -C(\mathrm{ch}, \mathrm{ch}')\widetilde{H} - u\widetilde{\gamma} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0. \tag{6.1.2}$$

Proof. Since M is a Mori dream space, we can identify the Néron-Severi groups by Example 3.4.3. The nef line bundle $\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')}$ is on a face of the rational polytope $g_i^*(\mathrm{Nef}(Y_i)) \times \mathrm{ex}(g_i)$ in the Mori chamber decomposition $\overline{\mathrm{Eff}}(M) = \bigcup_{i=1}^m g_i^*(\mathrm{Nef}(Y_i)) \times \mathrm{ex}(g_i)$ in Theorem C.2.4. $\square$

6.2 A conjecture on Hilbert scheme of points by Arcara, Bertram, Coskun, Huizenga

Arcara, Bertram, Coskun, Huizenga studied the Hilbert scheme of n-points on the projective plane $\mathbb{P}^2$ and gave a precise conjecture between the Bridgeland walls and Mori walls.

Conjecture 6.2.1. *[ABCH13, Conjecture 10.1] Let $S = \mathbb{P}^2$ and* $\mathrm{ch} = (1, 0, -n)$ *with integer $n \geq 2$. Let $h : S^{[n]} \to S^{(n)}$ be the Hilbert-Chow morphism, which maps the Hilbert scheme to the symmetric product. Denote* $\mathrm{H} := h^*(\mathcal{O}_{S^{(n)}}(1))$ *and* $\mathrm{B} := \{\xi \in S^{[n]} \mid |\mathrm{Supp}(\xi)| < n\}$ *the boundary divisor h. Let $x < 0$ be the center of a Bridgeland wall in the (s,t)-plane $\Pi_{(H,0,0)}$. Let $\mathrm{H} + \frac{1}{2y}\mathrm{B}$, $y < 0$, be a divisor class on $S^{[n]}$ spanning a wall of the Mori chamber decomposition. Arcara, Bertram, Coskun, Huizenga speculated that the transform $x = y - \frac{3}{2}$ gives a one-to-one correspondence between the two sets of walls for any integer $n \geq 2$ without any restriction on x and y.*

They [ABCH13] showed that the above speculation holds for $n \leq 9$ or when x and y are sufficiently small. There are some recent progresses by the work of Coskun, Huizenga, Woolf [CHW14] and the work of Coskun, Huizenga [CH14]. The speculation was further generalized to other rational surfaces by Bertram and Coskun [BC13]. We give a positive answer, which extends Problem 6.2.1 to more general invariants ch beside $(1, 0, -n)$.

Corollary 6.2.2. *(a). Let $S = \mathbb{P}^2$ and denote H the hyperplane divisor on $\mathbb{P}^2$. Fix* ch *with* $\mathrm{ch}_0 > 0$. *Assume that (i) there is a flat family $\mathcal{E}$; (ii) $M_{(\alpha,\omega)}(\mathrm{ch})$ is irreducible; (iii) $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$; (iv) $U_\omega(\mathrm{ch})$ is of positive dimension; (v) we can identify the Néron-Severi groups as Section 3.4. Then there is a relation*

$$\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} = -\left(C(\mathrm{ch},\mathrm{ch}') + \frac{3}{2}\right)\widetilde{H} - \mathcal{B}_0, \tag{6.2.1}$$

where $\widetilde{H}$ is the nef divisor on $M_{(\alpha,\omega)}(\mathrm{ch})$ which induced the Gieseker-Uhlenbeck morphism. If $\mathrm{ch}_0 = 1$ or 2, then $\mathcal{B}_0$ is the untwisted boundary divisor of the GU morphism.

(b). [LZ13, Theorem 0.2] In particular, if $\mathrm{ch} = (1, 0, -n)$*, then* $\widetilde{H} = \mathrm{H}$ *and* $\mathcal{B}_0 = \frac{1}{2}\mathrm{B}$.

The above assumptions are guaranteed if ch satisfies the condition (C) and $\mathrm{ch}_2 \ll 0$ by Remark 5.4.5.

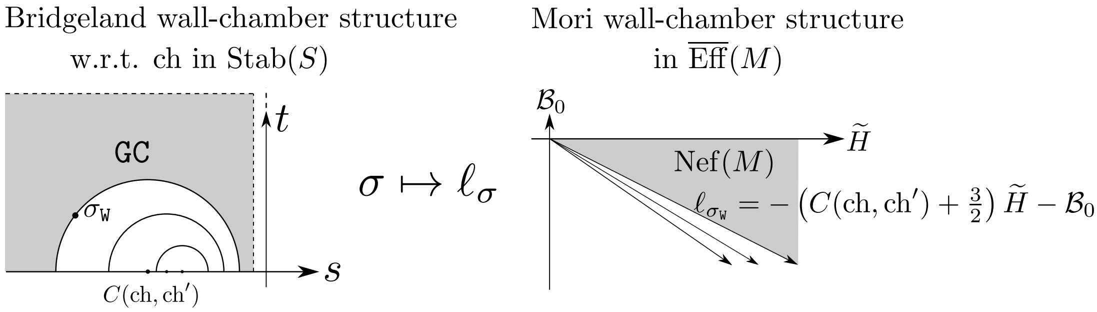


Figure 6.1: Bridgeland-Mori correspondence for $M = M_{(\alpha,\omega)}(\mathrm{ch})$ over $\mathbb{P}^2$

Proof. Recall that $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}H$ for the hyperplane divisor H and $K_{\mathbb{P}^2} \cong -3H$. So $\gamma = 0$. Moreover, $M_{(\alpha,H)}(\mathrm{ch})$ is a Mori dream space [BMW14]. The proof follows by using Corollary 6.1.4 and Example 5.4.7. $\square$

Remark 6.2.3. Li and Zhao [LZ13, Theorem 0.2] also solved the Problem 6.2.1 independently by using the GIT and "Quiver region" trick, which firstly appeared in [ABCH13]. Moreover, Li and Zhao extended the Problem 6.2.1 to the case where S is the non-commutative $\mathbb{P}^2$ in the sense of Sklyanin algebras.

Remark 6.2.4. Bertram and Coskun [BC13] studied other rational surfaces. The speculations therein can also be treated by applying Corollary 6.1.4 and Example 5.4.7.

6.3 A toy model: fibered surface over $\mathbb{P}^1$ with a section

Let $\pi : S \to \mathbb{P}^1$ be either a $\mathbb{P}^1$-fibered or an elliptic-fibered surface over $\mathbb{P}^1$ with a section E whose self-intersection number is $-e$. We *assume* that all fibers are reduced and irreducible. Denote the generic fiber class by F. We compute the nef cone of the Hilbert scheme $S^{[n]}$ of n-points over S by using the Theorem 6.1.1.

- $\mathbb{P}^1$ fibration. In this case, $F \cong \mathbb{P}^1$ and S is the *Hirzebruch surface* Σ_e with integer $e \geq 0$. Then $\mathrm{K}_S = -2E - (e+2)F$. It is well known that $\chi(\mathcal{O}_S) = 1$. Here Σ_0 is the surface $\mathbb{P}^1 \times \mathbb{P}^1$.
- Elliptic fibration. In this case, the generic fiber F is an elliptic curve. We denote the surface by S_e and further *assume* that $e \geq 2$. Then $\mathrm{K}_S = (e-2)F$. In particular, S_2 is an elliptic K3 surface, and the singular fibers are either cuspidal or nodal curves. Standard references are Miranda's lecture notes [Mir89] and Friedman's book [Fri98]. With the assumption $e \geq 2$, S_e has the unique section. Moreover, $\chi(\mathcal{O}_{S_e}) = e$, $p_g(S_e) = e - 1$ ([Fri98, Lemma 7.14]).

We exclude the surface S_1, which is a rational elliptic surface, and can be obtained by blowing up nine generic points of $\mathbb{P}^2$. There are infinite many sections in S_1.

The geometry of S

The Néron-Severi group $\mathrm{NS}(S)$ is generated by the section E and the fiber class F. We have the intersection number: $E.E = -e$, $E.F = 1$, $F^2 = 0$. The nef cone of the surface is generated by the two extremal nef line bundle $E + eF$ and F; and the ample cone is the interior of the nef cone.

Bridgeland Stability conditions

Fix $\mathrm{ch} = (1, 0, -n)$, with integer $n \geq 2$. Let us fix two numbers u and λ with $\lambda \in (0, 1)$. The data (λ, u) is regarded as the *initial value*. A Maciocia's coordinate (H, γ, u) is fixed by the choice:

$$H := \lambda(E + eF) + (1 - \lambda)F, \quad \gamma := -\lambda(E + eF) + (1 - \lambda + e\lambda)F \in H^{\perp}.$$

Basic computation shows that

$$H^2 = -\gamma^2 = \lambda^2 e + 2\lambda(1 - \lambda); \quad H.\gamma = 0.$$

The potential walls are given by $(s-C)^2+t^2=C^2+D$ with $t>0$ where

$$C=C(\mathrm{ch},\mathrm{ch}')=\frac{\mathrm{ch}_2'+\mathrm{ch}_0'n-u\mathrm{ch}_1'.\gamma}{\mathrm{ch}_1'.H},\quad D=D(\mathrm{ch},\mathrm{ch}')=-u^2-\frac{2n}{H^2}. \quad (6.3.1)$$

The number $s_0=\frac{\mathrm{ch}_1.H}{H^2}=0$, and the `UW` is given by $s=0$. Therefore $C<0$.

The geometry of $S^{[n]}$

One type of nef line bundle is $\widetilde{\omega}$ for $\omega\in\mathrm{Amp}(S)$, which induces Gieseker-Uhlenbeck morphism. Note that we can take ω to be extremal, i.e $\omega=E+eF$ or F. We then obtain two extremal nef line bundle $(\widetilde{E+eF})$, $\widetilde{F}$. In these cases, the Gieseker-Uhlenbeck morphism degenerates to some contractions. In the case of $\omega=F$, the fibration structure of S gives the fibration structure of $S^{[n]}$. In the case of $S_2^{[n]}$, it is the Lagrangian fibration.

To find the nef cone of $S^{[n]}$, we need to find the biggest non-trivial wall, i.e the smallest value of C.

As pointed out by Bertram and Coskun [BC13], we only consider the rank one walls, where $\mathrm{ch}'=(1,L,\frac{L^2}{2}-w)$ can be thought as the Chern character of $I_W\otimes L$, with W a 0-dimensional subscheme of length $w\geq 0$; L is a line bundle. Moreover, since we should consider the region $s<s_0$, we need $L.H<0$ to guarantee the line bundle L is a stable object in this region. Then $C(\mathrm{ch},\mathrm{ch}')=\frac{\frac{L^2}{2}+n-uL.\gamma}{L.H}+\frac{w}{-L.H}$. So we let $w=0$, and try to find the smallest value of $C(\mathrm{ch},\mathrm{ch}')=\frac{\frac{L^2}{2}+n-uL.\gamma}{L.H}$ for variables $\mathrm{ch}'=(1,L,\frac{L^2}{2})$ with the condition $L.H<0$.

Lemma 6.3.1. *The smallest value of* $C(\mathrm{ch},\mathrm{ch}')=\frac{\frac{L^2}{2}+n-uL.\gamma}{L.H}$ *is obtained by taking* $\mathrm{ch}'=(1,-F,0)$ *or* $\mathrm{ch}'=(1,-E,\frac{-e}{2})$.

Proof. We already know $\mathrm{ch}'=(1,-(mF+kE),\frac{(mF+kE).(mF+kE)}{2})$, where m and k are two non-negative integers, and $(m,k)\neq(0,0)$. Denote the line bundle corresponding to (m,k) by $\ell(m,k)$. The locus contracted by $\ell(0,1)$ is $\{Z\in S^{[n]}|\ Z\subset E\}$, which is isomorphic to $(\mathbb{P}^1)^{[n]}\cong\mathbb{P}^n$. The locus contracted by $\ell(1,0)$ is $\{Z\in S^{[n]}|\ Z\subset F,\ Z$ is linear equivalent to $n(E\cap F)\}$. In the case of $S=\Sigma_e$, the locus is a $\mathbb{P}^n$-bundle over $\Sigma\cong\mathbb{P}^1$. In the case of $S=S_e$, the locus is a $\mathbb{P}^{(n-1)}$-bundle over $\mathbb{P}^1$, which is given by $\mathbb{P}\big((\pi_*\mathcal{O}_S(nE))^*\big)$ [Mir89, (II. 4.3)]. In particular, $\pi_*\mathcal{O}_S(nE)\cong\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-4)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^1}(-2n)$. Assume that the smallest value is obtained by taking $(m,k) \neq (1,0)$ or $(0,1)$. Recall the walls are nested. But the locus contracted by $\ell(1,0)$ or $\ell(0,1)$ are also contracted by $\ell(m,k)$, which is a contradiction. $\square$

Denote the center $C(\mathrm{ch},\mathrm{ch}')$ for $\mathrm{ch}' = (1,-F,0)$ by C_F, and the center for $\mathrm{ch}' = (1,-E,\frac{-e}{2})$ by C_E. Then

$$C_F := \frac{n-u\lambda}{-\lambda}, \quad C_E := \frac{-\frac{e}{2}+n+u(1-\lambda+e\lambda)}{-(1-\lambda)}.$$

The two type of loci are simultaneous contracted if and only if $C_F = C_E$. By solving the equation, we obtain $(1-2\lambda)n + \lambda\frac{e}{2} = u\lambda(2-2\lambda+e\lambda)$. Therefore we have $\lambda = \frac{1}{2}$ and $u = \frac{e}{2+e}$. In this case, $C_F = C_E = -2n + \frac{e}{e+2}$. By using equation (6.1.1), we obtain the extremal nef line bundle:

$$\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} = n(\widetilde{E+eF}) + (n-\frac{e}{2})\widetilde{F} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0.$$

- In the case of $S = \Sigma_e$ with $e \geq 0$, $\mathrm{K}_S = -(2E+2F+eF)$, so the extremal nef line bundle is $\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} = (n-1)(\widetilde{E+eF}) + (n-1)\widetilde{F} - \frac{1}{2}\mathrm{B}$.
- In the case of $S = S_e$ with $e \geq 2$, $\mathrm{K}_S = (e-2)F$, so the extremal nef line bundle is $\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} = n(\widetilde{E+eF}) + (n-1)\widetilde{F} - \frac{1}{2}\mathrm{B}$.

Therefore, we have the following theorem.

Theorem 6.3.2. *(a). [BC13, Theorem 1 (2.) (3.)] The nef cone* $\mathrm{Nef}(\Sigma_e^{[n]})$ *(*$e \geq 0$, $n \geq 2$*) is generated by the non-negative combinations of* $(\widetilde{E+eF})$, $\widetilde{F}$, *and* $(n-1)(\widetilde{E+eF}) + (n-1)\widetilde{F} - \frac{1}{2}\mathrm{B}$.

(b). The nef cone $\mathrm{Nef}(S_e^{[n]})$ *(*$e \geq 2$, $n \geq 2$*) is generated by the non-negative combinations of* $(\widetilde{E+eF})$, $\widetilde{F}$, *and* $n(\widetilde{E+eF}) + (n-1)\widetilde{F} - \frac{1}{2}\mathrm{B}$.

Remark 6.3.3. The above computation suggests that the number u plays an important role in order to find the extremal nef line bundle, and in general $u \neq 0$. The nef cone of $\Sigma_e^{[n]}$ has been obtained by Bertram and Coskun [BC13]. The nef cone of $S_2^{[n]}$ $(n \geq 2)$ has been obtained by J. Li and W.-P. Li [LL10]. Both of the results are obtained using the notion of *k-very ample* line bundles [BSG91].

Remark 6.3.4. We recall a little geometry of $S_e^{[n]}$ by the work of J. Li and W.-P. Li [LL10]. The double covering $F \to \mathbb{P}^1$ induces an involution τ on F such that $F/\langle\tau\rangle \cong \mathbb{P}^1$. For any $n \geq 2$, the line C_n is defined to be the set

$$C_n := \{x + \tau(x) + (n-2)(E \cap F) |\, x \in F\} \subset \mathrm{Sym}^n(F) \subset S_e^{[n]}.$$

Given a curve C on S_e and distinct points $x_1, \ldots, x_{n-1} \in S$ with $x_i \notin C$, a curve β_C in $S_e^{[n]}$ is defined as

$$\beta_C := \{x + x_1 + \cdots + x_{n-1} |\, x \in C\}.$$

A curve β_n in $S_e^{[n]}$ is defined as

$$\beta_n := \{\xi + x_2 + \cdots + x_{n-1} |\, \xi \in S_e^{[2]},\ \mathrm{Supp}(\xi) = x \in S_e\}.$$

They showed that C_n is an extremal effective curve on $S_e^{[n]}$. Moreover, it is numerical equivalent to the curve $\beta_F - n\beta_n$. The following intersection number vanishes $\left(n(\widetilde{E + eF}) + (n-1)\widetilde{F} - \frac{1}{2}\mathrm{B}\right).C_n = 0$. This also follows from Theorem 3.1.1.

Appendix A

Twisted Gieseker stability and the large volume limit

Let us recall the Bridgeland stability conditions for $\omega^2 \gg 0$ [Bri08, Section 14].

Definition A.0.1. Fix Chern characters $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. A subset

$$\mathrm{LVL}(\mathrm{ch}) := \{\sigma_{\omega,\beta} \mid \omega \in \mathrm{Amp}(S), \omega^2 \gg 0\} \subset \mathrm{Stab}(S). \tag{A.0.1}$$

is called the *large volume limit for* ch. Define

$$\mathrm{LVL}(\mathrm{ch}, H) := \{\sigma_{\omega,\beta} \mid H \in \mathrm{Amp}(S) \text{ is fixed}, \omega = tH,\ t \gg 0\} \subset \mathrm{Stab}(S). \tag{A.0.2}$$

Twisted Gieseker stability was introduced independently (in equivalent form) by Ellingsrud-Göttsche [EG95], Friedman-Qin [FQ95], Matsuki-Wentworth [MW97].

Definition A.0.2. Let $\omega, \alpha \in \mathrm{NS}(S)_{\mathbb{Q}}$ with ω ample. For $E \in \mathrm{Coh}(S)$, we denote the leading coefficient of $\chi(E \otimes \alpha^{-1} \otimes \omega^{\otimes m})$ with respect to m by a_d. A coherent sheaf E of dimension d is said to be α-twisted ω-Gieseker-stable (or α-twisted ω-Gieseker-semistable respectively) if E is pure and for all $F \subsetneq E$ (for all $F \subset E$ respectively),

$$\frac{\chi(F \otimes \alpha^{-1} \otimes \omega^{\otimes m})}{a_d(F)} < (\leq \text{ respectively}) \frac{\chi(E \otimes \alpha^{-1} \otimes \omega^{\otimes m})}{a_d(E)} \quad \text{for } m \gg 0. \tag{A.0.3}$$

We shall use the brief notation (α, ω)-Gieseker (semi)stability for the α-twisted ω-Gieseker-(semi)stability.

Remark A.0.3. Fix an ample divisor H and take $\omega = tH$. It is easy to check (by the proof of Lemma A.0.5) that (α, tH)-Gieseker (semi)-stability is independent of the positive number t. Without loss of generality, we assume $t \gg 0$. Therefore, in the rest of the paper, when we use the notion (α, ω)-Gieseker (semi)stability, we *always assume* that $\omega^2 \gg 0$.

By the Hirzebruch-Riemann-Roch theorem, we obtain

$$\begin{aligned} \chi(E \otimes \alpha^{-1} \otimes \omega^{\otimes m}) &= \frac{1}{2}\mathrm{ch}_0 \omega^2 m^2 + \omega.(\mathrm{ch}_1 - \mathrm{ch}_0 \beta)\, m + \mathrm{ch}_2 - \mathrm{ch}_1.\beta \\ & -\frac{1}{2}\mathrm{ch}_0 \left(\frac{1}{4}\mathrm{K}_S^2 - 2\chi(\mathcal{O}_S) - \beta^2 \right), \end{aligned} \tag{A.0.4}$$

where $\beta = \alpha + \frac{1}{2}\mathrm{K}_S$, and we use the short notation $\mathrm{ch}_i = \mathrm{ch}_i(E)$.

Notation A.0.4. Let us denote

$$\xi_{\omega,\beta}(E) := \frac{\mathrm{ch}_2(E) - \mathrm{ch}_1(E).\beta}{\omega.\mathrm{ch}_1(E)}, \quad \nu_\beta(E) := \frac{\mathrm{ch}_2(E) - \mathrm{ch}_1(E).\beta}{\mathrm{ch}_0(E)}. \tag{A.0.5}$$

Lemma A.0.5. *Choose an ample line bundle ω with $\omega^2 \gg 0$ and take*

$$\alpha := \beta - \frac{1}{2}\mathrm{K}_S, \tag{A.0.6}$$

then for $E \in \mathcal{A}_{\omega,\beta} \cap \mathrm{Coh}(S) = \mathcal{T}_{\omega,\beta}$, we have the equivalent relation (A.0.3) $\Longleftrightarrow$ (2.1.3), i.e. (α, ω)-Gieseker (semi)stability is exactly the same as $\sigma_{\omega,\beta}$-Bridgeland (semi)stablility for $E \in \mathcal{T}_{\omega,\beta}$ with $\omega^2 \gg 0$.

Proof. The lemma is well-known (e.g. [Bri07, Section 14]). We sketch the proof. We still use the short notation $\mathrm{ch}_i = \mathrm{ch}_i(E)$. Since $E \in \mathcal{T}_{\omega,\beta}$, we know $\mathrm{ch}_0 \geq 0$. Let us consider (α, ω)-Gieseker (semi)stability. If $\mathrm{ch}_0 = 0$, they are known as Simpson (semi)stability.

Case 1: $\mathrm{ch}_0 = 0$ and $\omega.\mathrm{ch}_1 = 0$. In this case, E is supported on points, so we have $a_d = a_0$, and $\frac{\chi(E \otimes \alpha^{-1} \otimes \omega^{\otimes m})}{a_d(E)} = \frac{a_0}{a_0} = 1$. So E is (α, ω)-Gieseker semistable (for any α).

Case 2: $\mathrm{ch}_0 = 0$ and $\omega.\mathrm{ch}_1 \neq 0$. Since $E \in \mathcal{T}_{\omega,\beta}$ we have $a_d = a_1 = \omega.\mathrm{ch}_1 > 0$, and

$$\frac{\chi(E \otimes \alpha^{-1} \otimes \omega^{\otimes m})}{a_d(E)} - m = \frac{\mathrm{ch}_2 - \mathrm{ch}_1.\beta}{\omega.\mathrm{ch}_1} = \xi_{\omega,\beta}(E).$$

Case 3: $\mathrm{ch}_0 > 0$. In this case we have $a_d = a_2 = \frac{1}{2}\mathrm{ch}_0\omega^2$. Recall the Mumford slope $\mu_\omega(E)$ in equation (2.1.4). Then by using equation (A.0.4), the relation (A.0.3) is equivalent to
Subcase 3.a: $\mu_\omega(F) < \mu_\omega(E)$; or
Subcase 3.b: $\mu_\omega(F) = \mu_\omega(E)$ and $\nu_\beta(F) < (\leq)\nu_\beta(E)$.

Let us consider Bridgeland (semi)stability.

Case 1: $\mathrm{ch}_0 = 0$ and $\omega.\mathrm{ch}_1 = 0$. In this case, E is supported on points so $\mathrm{ch}_2 > 0$ and $\phi(E) = 1$. So E is Bridgeland semistable (for any β).

Case 2: $\mathrm{ch}_0 = 0$ and $\omega.\mathrm{ch}_1 \neq 0$. Since $E \in \mathcal{T}_{\omega,\beta}$, we know $\omega.\mathrm{ch}_1 > 0$. We obtain $\mu_\sigma(E) = \xi_{\omega,\beta}(E)$ by using equations (2.1.2) and (2.0.1).

Case 3: $\mathrm{ch}_0 > 0$. Since $E \in \mathcal{T}$, we obtain $\Im Z > 0$, i.e, $\mu_\omega(E) - \eta > 0$, where $\eta = \omega.\beta$ as equation (2.1.5). By using equation (2.1.2), we obtain

$$\frac{Z(E)}{\mathrm{ch}_0(E)} = \frac{1}{2}\left(\omega^2 - \beta^2\right) - \nu_\beta(E) + \sqrt{-1}(\mu_\omega(E) - \eta).$$

Now relation (2.1.3) is equivalent to $\Im(E)\Re(F) - \Im(F)\Re(E) > (\geq)0$. This relation is equivalent to

$$\begin{aligned}&(\mu_\omega(E) - \eta)\left(\frac{1}{2}\left(\omega^2 - \beta^2\right) - \nu_\beta(F)\right) - \\ &(\mu_\omega(F) - \eta)\left(\frac{1}{2}\left(\omega^2 - \beta^2\right) - \nu_\beta(E)\right) > (\geq)0. \qquad \text{(A.0.7)}\end{aligned}$$

Since ω is ample and $\omega^2 \gg 0$, we consider coefficient of ω^2 in formula (A.0.7) and obtain two subcases:
Subcase 3.a: $(\eta <)\ \mu_\omega(F) < \mu_\omega(E)$;
Subcase 3.b: $(\eta <)\ \mu_\omega(F) = \mu_\omega(E)$ and $\nu_\beta(F) < (\leq)\nu_\beta(E)$.

□

Remark A.0.6. Let $E \in \mathcal{A}_{\omega,\beta}$ be a $\sigma_{\omega,\beta}$-semistable object with $\mathrm{ch}_0(E) \geq 0$. Assume that $\omega^2 \gg 0$. Bridgeland [Bri08, Proposition 14.2] showed that E is a sheaf of pure dimension, i.e. $H^{-1}(E) = 0$ and $E \cong H^0(E)$ is a sheaf of *pure* dimension ([HL10, Definition 1.1.2]).

Remark A.0.7. Recall that there is a wall-chamber structure in $\mathrm{Stab}(S)$ for the fixed ch (e.g. [BM14a, Proposition 2.3]). Assume that $\mathrm{ch}_0 \geq 0$. Recall that we always assume $\omega^2 \gg 0$ as in Remark A.0.3. Therefore the classical wall-chamber structure ([EG95, FQ95, MW97]) for the (α, ω)-Gieseker (semi)stability is a special case of wall-chamber structure for the $\sigma_{\omega,\beta}$-Bridgeland (semi)stability in the large volume limit for ch (with the twisted relation (A.0.6)). Yoshioka [Yos14, Section 5] obtained the same results.

Appendix B

Bayer-Macrì decomposition on K3 surfaces by using $\hat{\sigma}_{\omega,\beta}$

The reason for using $\hat{Z}_{\omega,\beta}$ on K3 surfaces is that, with some assuumptions, $\hat{\sigma}_{\omega,\beta}$ is a *reduced* stability condition (Remark 2.4.3). We first give some computations without using $\mathrm{K}_S = 0$ in Section B.1. We then give the decomposition in Section B.2.

B.1 Computation by using $\mho_{\hat{Z}}$

Recall that $\hat{\sigma}_{\omega,\beta} := (\hat{Z}_{\omega,\beta}, \mathcal{A}_{\omega,\beta+\frac{1}{4}\mathrm{K}_S})$. Simply write $\hat{Z}_{\omega,\beta}$, $\hat{\sigma}_{\omega,\beta}$, $\mho_{\hat{Z}_{\omega,\beta}}$ as $\hat{Z}$, $\hat{\sigma}$, $\mho_{\hat{Z}}$ respectively. Recall $\hat{w}_{\omega,\beta} := \hat{w}_{\hat{\sigma}} := -\Im\left(\overline{\langle \mho_{\hat{Z}}, \mathbf{v}\rangle_S} \cdot \mho_{\hat{Z}}\right)$. In this Appendix, we compute the Bayer-Macrì decomposition of $\hat{w}_{\omega,\beta}$. By the definition equation (2.1.6) and Lemma 2.4.1, we have

$$\hat{Z}_{\omega,\beta}(E) = Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S} + \frac{1}{24}\mathrm{ch}_0(E)\mathrm{ch}_2(S), \tag{B.1.1}$$

$$\mho_{\hat{Z}_{\omega,\beta}} = \mho_{Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}} - (0, 0, \frac{1}{24}\mathrm{ch}_2(S)). \tag{B.1.2}$$

So

$$\mu_{\hat{\sigma}_{\omega,\beta}}(\mathrm{ch}) = -\frac{\Re \hat{Z}_{\omega,\beta}}{\Im \hat{Z}_{\omega,\beta}} = \mu_{\sigma_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}}(\mathrm{ch}) - \frac{\frac{1}{24}\mathrm{ch}_0(E)\mathrm{ch}_2(S)}{\Im Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}}. \tag{B.1.3}$$

Similar as Lemma 3.2.2, if $\Im Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S} \neq 0$, we have

$$\hat{w}_{\omega,\beta} \overset{\mathbb{R}_+}{=\!=} \mu_{\hat{\sigma}}(\mathrm{ch})\Im\mho_{\hat{Z}} + \Re\mho_{\hat{Z}}. \tag{B.1.4}$$

Next, let us compute the potential walls $\hat{W}(\mathrm{ch}, \mathrm{ch}')$ as in Section 2.5. Fix the triple (H, γ, u), where H is an ample divisor, γ is another divisor such that $\gamma \in H^{\perp}$, u is a real number. We take a *different* Maciocia's coordinate (comparing with equation (2.5.1)), for real numbers s, t, with t positive:

$$\begin{cases} \omega := tH, \\ \beta + \frac{1}{4}\mathrm{K}_S := sH + u\gamma. \end{cases} \tag{B.1.5}$$

Take ch and ch$'$ as before (equations (1.5.3, 1.5.4)). By solving the equation $\mu_{\hat{Z}_{\omega,\beta}}(\mathrm{ch}') = \mu_{\hat{Z}_{\omega,\beta}}(\mathrm{ch})$ for variables s and t, we obtain the potential wall $\hat{W}(\mathrm{ch}, \mathrm{ch}')$ in the (s,t)-half-plane, which is still semicircle

$$(s - \hat{C})^2 + t^2 = \hat{C}^2 + \hat{D}, \quad t > 0, \tag{B.1.6}$$

where

$$\hat{C} = C, \quad \hat{D} = D - \frac{\mathrm{ch}_2(S)}{12g}, \tag{B.1.7}$$

and C, D are given in Theorem 2.5.2.

Moreover, we still have similar formulas as equations (5.1.1, 5.1.2):

$$\mu_{\hat{\sigma}_{\omega,\beta}}(\mathrm{ch})\omega + \beta + \frac{1}{4}\mathrm{K}_S = \hat{C}H + u\gamma; \tag{B.1.8}$$

$$(\beta + \frac{1}{4}\mathrm{K}_S).[\mu_{\hat{\sigma}_{\omega,\beta}}(\mathrm{ch})\omega + \beta + \frac{1}{4}\mathrm{K}_S] - \frac{1}{2}(\omega^2 + (\beta + \frac{1}{4}\mathrm{K}_S)^2) = -\frac{g}{2}\hat{D} - \frac{d}{2}u^2. \tag{B.1.9}$$

Therefore, if $\Im Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}(\mathrm{ch}) \neq 0$, by using equation (B.1.4) and direct computation similar to equation (5.1.4), we have

$$\hat{w}_{\sigma \in \hat{W}(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=} \left(\frac{g}{2}D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2}u^2\right)(0, 0, -1) + \mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}, \mathrm{ch}'). \tag{B.1.10}$$

- If $\mathrm{ch}_0 = 0$ then both $(0, 0, -1)$ and $\mathbf{t}_{(H,\gamma,u)}(0, \mathrm{ch}_0, \mathrm{ch}_1)$ are in $\mathbf{v}^{\perp}$ and the decomposition B.1.10 is the Bayer-Macrì decomposition of $\hat{w}_{\sigma \in \hat{W}(\mathrm{ch},\mathrm{ch}')}$.
- If $\mathrm{ch}_0 \neq 0$ and $\Im Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}(\mathrm{ch}) \neq 0$, by using the above relations (B.1.1, B.1.2, B.1.3), we have

  $$\hat{w}_{\omega,\beta} \overset{\mathbb{R}_+}{=\!=} w_{\omega,\beta+\frac{1}{4}\mathrm{K}_S} - (0, 0, \frac{1}{24}\mathrm{ch}_2(S)) - \frac{\frac{1}{24}\mathrm{ch}_0(E)\mathrm{ch}_2(S)}{\Im Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}} \Im \mho_{Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}}.$$

  Then by the decomposition formula (5.2.4) we have

  $$\begin{aligned} w_{\omega,\beta+\frac{1}{4}\mathrm{K}_S} &= \mu_{\sigma_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}}(\mathrm{ch})\mathbf{m}(\omega, \mathrm{ch}) + \mathbf{m}(\beta + \frac{1}{4}\mathrm{K}_S, \mathrm{ch}) + \mathbf{w}(\mathrm{ch}) \\ &= \mu_{\hat{\sigma}_{\omega,\beta}}(\mathrm{ch})\mathbf{m}(\omega, \mathrm{ch}) + \mathbf{m}(\beta + \frac{1}{4}\mathrm{K}_S, \mathrm{ch}) + \mathbf{w}(\mathrm{ch}) \\ &\quad + \frac{\frac{1}{24}\mathrm{ch}_0(E)\mathrm{ch}_2(S)}{\Im Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}}\mathbf{m}(\omega, \mathrm{ch}). \end{aligned}$$

A direct computation shows that

$$\frac{\frac{1}{24}\mathrm{ch}_0(E)\mathrm{ch}_2(S)}{\Im Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}}\left(\mathbf{m}(\omega,\mathrm{ch})-\Im\mho_{Z_{\omega,\beta+\frac{1}{4}\mathrm{K}_S}}\right)=(0,0,\frac{1}{24}\mathrm{ch}_2(S)).$$

Therefore we obtain the Bayer-Macrì decomposition

$$\hat{w}_{\omega,\beta}=\mu_{\hat{\sigma}_{\omega,\beta}}(\mathrm{ch})\mathbf{m}(\omega,\mathrm{ch})+\mathbf{m}(\beta+\frac{1}{4}\mathrm{K}_S,\mathrm{ch})+\mathbf{w}(\mathrm{ch}). \tag{B.1.11}$$

B.2 Bayer-Macrì decomposition on K3 surfaces

In the following, we assume that S is a smooth projective K3 surface. Take $H,\gamma\in\mathrm{NS}(S)$ with H ample and $H.\gamma=0$. Since $\mathrm{K}_S=0$, the new Maciocia's coordinate (B.1.5) is the same as before (2.5.1): $\omega=tH$ $(t>0)$, $\beta=sH+u\gamma$. Assume that $\hat{Z}_{\omega,\beta}(F)\notin\mathbb{R}_{\leq 0}$ for all spherical sheaves $F\in\mathrm{Coh}(S)$. Then $\hat{\sigma}_{\omega,\beta}=(\hat{Z}_{\omega,\beta},\mathcal{A}_{\omega,\beta})$ is a Bridgeland stability condition. Define

$$\ell_{\hat{\sigma},\mathcal{E}}([C])=\ell_{\hat{\sigma},\mathcal{E}}.C:=\Im\left(-\frac{\hat{Z}\left(\Phi_{\mathcal{E}}(\mathcal{O}_C)\right)}{\hat{Z}(\mathrm{ch})}\right). \tag{B.2.1}$$

Then $\ell_{\hat{\sigma},\mathcal{E}}$ (if exists) is given by

$$\ell_{\hat{\sigma}_{\omega,\beta}}\overset{\mathbb{R}_+}{=\!=}\theta_{\hat{\sigma},\mathcal{E}}(\hat{w}_{\omega,\beta}).$$

The potential walls $\hat{W}(\mathrm{ch},\mathrm{ch}')$ are given by semicircles

$$(s-C)^2+t^2=C^2+D+\frac{2}{g},$$

where C and D are defined in Theorem 2.5.2.

Theorem B.2.1. *(Bayer-Macrì decomposition on K3 surfaces.) Let S be a smooth projective K3 surface and $\mathbf{v}=v(\mathrm{ch})\in H^*_{\mathrm{alg}}(S,\mathbb{Z})$ be a primitive class with $\langle\mathbf{v},\mathbf{v}\rangle_S>0$. Assume that $\hat{Z}_{\omega,\beta}(F)\notin\mathbb{R}_{\leq 0}$ for all spherical sheaves $F\in\mathrm{Coh}(S)$.*

- *The case $\mathrm{ch}=(0,\mathrm{ch}_1,\mathrm{ch}_2)$ with $\mathrm{ch}_1.H>0$ for a choice of Maciocia's coordinate (H,γ,u). Then $\hat{w}_{\omega,\beta}$ has a decomposition*

$$\hat{w}_{\sigma\in\hat{W}(\mathrm{ch},\mathrm{ch}')}\overset{\mathbb{R}_+}{=\!=}\left(\frac{g}{2}D(\mathrm{ch},\mathrm{ch}')+\frac{d}{2}u^2\right)(0,0,-1)+\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}), \tag{B.2.2}$$

and the coefficient before $(0,0,-1)$ *is given by equation (*5.2.2*). Moreover, the Bayer-Macrì line bundle has a decomposition*

$$\ell_{\hat{\sigma}\in\hat{W}(\mathrm{ch},\mathrm{ch}')} = \left(\frac{g}{2}D(\mathrm{ch},\mathrm{ch}') + \frac{d}{2}u^2\right)\mathcal{S} + \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}), \tag{B.2.3}$$

where $\mathcal{S}$ *induces the support morphism.*

- *The case* $\mathrm{ch}_0 > 0$*. Assume further* $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$*. Then*

$$\hat{w}_{\omega,\beta} = \mu_{\hat{\sigma}_{\omega,\beta}}(\mathrm{ch})\mathbf{m}(\omega,\mathrm{ch}) + \mathbf{m}(\beta,\mathrm{ch}) + \mathbf{w}(\mathrm{ch}). \tag{B.2.4}$$

Moreover, the Bayer-Macrì line bundle has a decomposition

$$\ell_{\hat{w}_{\omega,\beta}} = (-\mu_{\hat{\sigma}_{\omega,\beta}}(\mathrm{ch}))\widetilde{\omega} - \widetilde{\beta} - \mathcal{B}_0. \tag{B.2.5}$$

Fix a Maciocia's coordinate (H,γ,u)*. There is a Bridgeland-Mori correspondence, which maps a Bridgeland wall* $\hat{W}(\mathrm{ch},\mathrm{ch}')$ *with center* $(C,0)$ *in the plane* $\Pi_{(H,\gamma,u)}$ *to the line bundle on the Mori wall*

$$\ell_{\hat{\sigma}\in\hat{W}(\mathrm{ch},\mathrm{ch}')} = -C\widetilde{H} - u\widetilde{\gamma} - \mathcal{B}_0. \tag{B.2.6}$$

The line bundle $\widetilde{\omega}$ *(or* $\widetilde{H}$*) induces the Gieseker-Uhlenbeck morphism.*

Proof. With the assumption on $\hat{Z}_{\omega,\beta}(F) \notin \mathbb{R}_{\leq 0}$ for all spherical sheaves F and by Bridgeland's Lemma 2.1.6, $\hat{\sigma}_{\omega,\beta}$ is a reduced Bridgeland stability condition. Since $\mathbf{v} = v(\mathrm{ch})$ is a primitive class, the moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ exists and is irreducible of dimension $\langle \mathbf{v},\mathbf{v}\rangle_S + 2$. Moreover, we have the global Bayer-Macrì map (Theorem 3.4.2). We can identify the Néron-Severi groups as Example 3.4.1.

In the case $\mathrm{ch}_0 = 0$ and $\mathrm{ch}_1.H > 0$, by Example 4.6.2 we know $\mathcal{S}$ induces the support morphism.

In the case $\mathrm{ch}_0 > 0$, since $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$ and

$$\mathrm{codim}(\partial M_{(\alpha,\omega)}(\mathrm{ch}), M_{(\alpha,\omega)}(\mathrm{ch})) \leq \mathrm{ch}_0 - 1,$$

we have $U_\omega(\mathrm{ch}) \neq \emptyset$. The Gieseker-Uhlenbeck morphism is not an isomorphism, and that is induced by $\widetilde{H}$. The remaining parts are clear. □

Appendix C

Divisors, cones and Mori dream spaces

We work over the complex number $\mathbb{C}$.

C.1 Divisors and cones

We collect some facts and notations on divisors. One can refer [Laz04] for details. Denote by $\mathcal{M}_X$ the sheaf of total quotient rings of $\mathcal{O}_X$, which contains the structure sheaf $\mathcal{O}_X$ as a subsheaf. There is an inclusion $\mathcal{O}_X^* \subset \mathcal{M}_X^*$ of sheaves of multiplicative abelian groups of units. A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. We simply call a Cartier divisor as a *divisor*. Denote $\mathrm{Div}(X) := \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$. Concretely, a divisor $D \in \mathrm{Div}(X)$ is represented by data $\{(U_i, f_i)\}$ consisting of an open cover $\{U_i\}$ of X together with elements $f_i \in \Gamma(U_i, \mathcal{M}_X^*)$, with the property that on $U_{ij} := U_i \cap U_j$ one can write $f_i = g_{ij} f_j$ for some $g_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^*)$. A divisor D is called *effective* if $f_i \in \Gamma(U_i, \mathcal{O}_X)$ is regular on U_i. A divisor $D \in \mathrm{Div}(X)$ determines a line bundle $\mathcal{O}_X(D)$ on X via the canonical connecting homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X) := H^1(X, \mathcal{O}_X^*)$. By abuse of notation, we do not distinguish a Cartier divisor D from its associated line bundle $\mathcal{O}_X(D)$. A $\mathbb{Q}$-*divisor* (respectively $\mathbb{R}$-*divisor*) is a finite $\mathbb{Q}$-linear (respectively $\mathbb{R}$-linear) combination of integral Cartier divisors. We denote $\mathrm{Div}_{\mathbb{Q}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathrm{Div}_{\mathbb{R}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. A *Weil* *divisor* is an $(n-1)$-cycle, i.e. a formal sum of codimension one subvarieties with integer coefficients. A normal variety X is called $\mathbb{Q}$-*factorial* if every Weil divisor is a $\mathbb{Q}$-divisor.

Two divisors D_1 and D_2 are called *numerically equivalent* if $D_1.C = D_2.C$ for every irreducible curve C in X. Denote $\mathrm{Num}(X) \subset \mathrm{Div}(X)$ the subgroup consisting of all divisors which are numerically equivalent to zero. Define the group $N^1(X)_{\mathbb{Z}} := \mathrm{Div}(X)/\mathrm{Num}(X)$ as the group of *numerical equivalence classes* of divisors on X. Denote $\mathrm{Alg}(X) \subset \mathrm{Div}(X)$ the subgroup consisting of all divisors which are *algebraic equivalent* to zero [Ful98, Definition 10.3]. Define the *Néron-Severi groups* $\mathrm{NS}(X) := \mathrm{Div}(X)/\mathrm{Alg}(X)$ as the group of *algebraic equivalence classes* of divisors on X. In particular, algebraic equivalence implies numerical equivalence. There is a well-defined *intersection number theory* among numerical equivalence classes [Laz04, Definition 1.1.19]. The Néron-Severi Theorem (Theorem of the base) states that $\mathrm{NS}(X)$ and $N^1(X)_{\mathbb{Z}}$ are finitely generated groups, and $N^1(X)_{\mathbb{Z}}$ is free. Moreover, if X is non-singular, then [Ful98, 19.3.1] $\mathrm{Num}(X)/\mathrm{Alg}(X) = H^2(X,\mathbb{Z})_{\mathrm{tors}}$ is a finite group. We have the isomorphisms

$$N^1(X)_{\mathbb{Q}} := N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{NS}(X)_{\mathbb{Q}}, \quad N^1(X) := N^1(X)_{\mathbb{R}} \cong \mathrm{NS}(X)_{\mathbb{R}}, \tag{C.1.1}$$

and their dimension (or the rank of $\mathrm{NS}(X)$) is called the *Picard number* of X, written by $\rho(X)$. Elements inside $N^1(X)_{\mathbb{Q}}$ or $N^1(X)_{\mathbb{R}}$ are called $\mathbb{Q}$-*divisor classes* or $\mathbb{R}$-*divisor classes* respectively.

A line bundle L on X is called *very ample* if $L = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$ for some embedding $\phi : X \hookrightarrow \mathbb{P}^n$. A line bundle L is called *ample* if $L^{\otimes m}$ is very ample for some integer $m > 0$. A $\mathbb{Q}$-Cartier divisor D on X is ample if the line bundle $\mathcal{O}_X(pD)$ is ample where pD is an integral divisor for some positive integer p. A divisor D is called *nef* if $D.C \geq 0$ for every irreducible curve $C \subset X$. Being ample is a *numerical property*, i.e. if D_1 and D_2 are numerically equivalent divisors, then D_1 is ample if and only if D_2 is ample. Being nef is also a numerical property. The numerical classes of ample divisors form an open convex cone, called the ample cone $\mathrm{Amp}(X)$ in $N^1(X)$. The numerical classes of nef divisors form an closed convex cone, called the nef cone $\mathrm{Nef}(X)$ in $N^1(X)$. Moreover, by Kleiman's Theorem [Laz04, Theorem 1.4.9], it is known that the ample cone is the interior of the nef cone and the nef cone is the closure of the ample cone:

$$\mathrm{Amp}(X) = \mathrm{int}\left(\mathrm{Nef}(X)\right), \quad \mathrm{Nef}(X) = \overline{\mathrm{Amp}}(X).$$

Let L be a line bundle and $V \subseteq H^0(X, L)$ a non-zero subspace of finite dimension. Denote by $|V| := \mathbb{P}_{\mathrm{sub}}(V)$ the projective space of one-dimensional subspaces of V. If $V = H^0(X, L)$, we write $|L| := |V|$ and call it a *complete linear system*. We also write $|D| := |\mathcal{O}_X(D)|$. Evaluation of sections in V gives rise to a morphism $\mathrm{eval}_V : V \otimes_{\mathbb{C}} \mathcal{O}_X \to L$. The *base ideal* of $|V|$, denoted by $\mathfrak{b}(|V|) \subseteq \mathcal{O}_X$, is the image of the map $V \otimes_{\mathbb{C}} L^* \to \mathcal{O}_X$ determined by eval_V. The *base locus* $\mathrm{Bs}(|V|) \subseteq X$ of $|V|$ is the closed subset of X cut out by the base ideal $\mathfrak{b}(|V|)$. In other words, $\mathrm{Bs}(|V|)$ is the set of points at which all the sections in V vanish, and $\mathfrak{b}(|V|)$ is the ideal sheaf spanned by these sections. One say that $|V|$ is *base-point-free*, if $\mathrm{Bs}(|V|) = \emptyset$, i.e. if $\mathfrak{b}(|V|) = \mathcal{O}_X$. A divisor D or a line bundle L is base-point-free if $|D|$ or $|L|$ is so. One says synonymously that L is *generated by its global sections* or *globally generated* if L is base-point-free. A line bundle L is called *semi-ample* if $L^{\otimes m}$ is globally generated for some integer $m > 0$. $|V|$ determines a morphism

$$\phi_{|V|} : X - \mathrm{Bs}(|V|) \longrightarrow \mathbb{P}(V).$$

We view $\phi_{|V|}$ as a rational map $\phi_{|V|} : X \dashrightarrow \mathbb{P}(V)$.

The semi-group $N(X, L)$ is defined as $N(X, L) := \{m \geq 0 \mid H^0(X, L^{\otimes m}) \neq 0\}$. In particular $N(X, L) = 0$ if $H^0(X, L^{\otimes m}) = 0$ for all $m > 0$. Given $m \in N(X, L)$ and assume $m \neq 0$, there is a rational map

$$\phi_m = \phi_{|L^{\otimes m}|} : X \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes m})).$$

The closure of its image is denoted by $Y_m = \phi_m(X) \subseteq \mathbb{P}(H^0(X, L^{\otimes m}))$. Assume that X is normal. The *Iitaka dimension* of L is defined to be

$$\kappa(X, L) := \max_{m \in N(X,L)} \{\dim \phi_m(X)\},$$

if $N(X, L) \neq 0$. $\kappa(X, L) := -\infty$ if $N(X, L) = 0$. If X is non-normal, define $\kappa(X, L) := \kappa(X', \nu^* L)$ by passing to its normalization $\nu : X' \to X$. It is known that $\kappa(X, L) = -\infty$ or $0 \leq \kappa(X, L) \leq \dim X$. A line bundle L is called *big* if $\kappa(X, L) = \dim X$. A divisor D is called big if $\mathcal{O}_X(D)$ is big. A $\mathbb{Q}$-divisor is called big (respectively effective) if there is a positive integer p such that pD is integral and big (respectively effective). A $\mathbb{R}$-divisor D is called big (respectively effective) if $D = \Sigma a_i \cdot D_i$ where each D_i is a big (respectively effective) integral divisor and a_i is a positive real number. Being big (respectively effective) is also a numerical property. The numerical equivalent classes of big $\mathbb{R}$-divisors form an open convex cone, called the big cone $\mathrm{Big}(X)$ in $N^1(X)$. The *pseudo-effective cone* $\overline{\mathrm{Eff}}(X)$ is defined as the closure of the convex cone spanned by the numerical equivalent classes of all effective $\mathbb{R}$-Cartier divisors. It is known that the big cone is the interior of the pseudo-effective cone and the pseudo-effective cone is the closure of the big cone [Laz04, Theorem 2.2.26]

$$\mathrm{Big}(X) = \mathrm{int}\left(\overline{\mathrm{Eff}}(X)\right), \quad \overline{\mathrm{Eff}}(X) = \overline{\mathrm{Big}}(X).$$

The *stable base locus* of an integral divisor D is the algebraic set

$$\mathbf{B}(D) = \bigcap_{m \geq 1} \mathrm{Bs}(|mD|).$$

The base loci $\mathrm{Bs}(|mD|)$ stabilize to $\mathbf{B}(D)$ for sufficiently large and divisible m [Laz04, Proposition 2.1.21], i.e. there exists a positive integer m_0 such that

$$\mathbf{B}(D) = \mathrm{Bs}(|km_0D|) \text{ for all } k \gg 0.$$

For any Cartier divisor D with $\kappa(X, D) \geq 0$ one has $\mathbf{B}(pD) = \mathbf{B}(D)$ for all $p \geq 1$. So for any $\mathbb{Q}$-divisor D, we define $\mathbf{B}(D) := \mathbf{B}(pD)$ where pD is integral. A divisor D is called *movable* if $\mathbf{B}(D)$ has codimension ≥ 2, i.e. $\mathbf{B}(D)$ has no divisorial component. The real cone generated by movable divisors is denoted by $\mathrm{Mov}(M)$. There are some relations of different cones:

$$\overline{\mathrm{Amp}}(X) = \mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}}(X) \subseteq \overline{\mathrm{Big}}(X) = \overline{\mathrm{Eff}}(X) \subset N^1(X).$$

However, the stable base locus is not a numerical invariant. It is possible that two divisors D_1 and D_2 are numerical equivalent, but $\mathbf{B}(D_1) \neq \mathbf{B}(D_2)$. With some assumptions and finer notions (e.g. augmented base locus and restricted base locus), the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$ can be divided into different kinds of wall-chambers [ELMNP06, ELMNP09, Cho15]. To give a precise statement about the decomposition of the pseudo-effective cone (Theorem C.2.4), we focus on the Mori dream spaces, which were introduced by Hu and Keel [HK00].

C.2 Mori dream spaces

Besides [HK00], we also refer the expository paper by J. McKernan [McK10].

Definition C.2.1. A normal projective variety M is called a *Mori dream space* if the following three conditions hold:

(a). M is $\mathbb{Q}$-factorial and $\mathrm{Pic}(M)_{\mathbb{Q}} = \mathrm{NS}(M)_{\mathbb{Q}}$;

(b). $\mathrm{Nef}(M)$ is the affine hull of finitely many semi-ample line bundles; and

(c). there are finitely $1 \leq i \leq k$ many SQMs $f_i : M \dashrightarrow M_i$ such that each M_i satisfies (b) and $\mathrm{Mov}(M)$ is the union of the $f_i^*(\mathrm{Nef}(M_i))$.

In above (c), by a *small $\mathbb{Q}$-factorial modification* (SQM) of a projective variety M we mean a contracting birational map $f : M \dashrightarrow M'$, with M' projective and $\mathbb{Q}$-factorial, such that f is an isomorphism in codimension one. Therefore, if D is a movable divisor on M then there is an index i and an semi-ample divisor D_i on M_i such that $D = f_i^* D_i$. Then the Néron-Severi groups of different birational models M_i can be identified.

Let $D_1, D_2, \ldots, D_k$ be a sequence of divisors on M. The *multi-graded section ring* associated to them is the ring

$$R(M, D_1, D_2, \ldots, D_k) := \bigoplus_{m \in \mathbb{Z}^k} H^0(M, \mathcal{O}_M(D)) \text{ where } D = \sum m_i D_i.$$

We call $R(M, D_1)$ a *section ring* when $k = 1$. Suppose $\mathrm{Pic}(M)$ is a finitely generated abelian group. Pick a set of divisors $D_1, D_2, \ldots, D_k$ so that the line bundles $\mathcal{O}_M(D_1), \mathcal{O}_M(D_2), \ldots, \mathcal{O}_M(D_k)$ generate $\mathrm{Pic}(M)$. Then the *Cox ring* $\mathrm{Cox}(M)$ (also called the total coordinate ring) is defined by $\mathrm{Cox}(M) := R(M, D_1, D_2, \ldots, D_k)$.

Lemma C.2.2. *[HK00, Proposition 2.9] Let M be a $\mathbb{Q}$-factorial projective variety over $\mathbb{C}$ such that $\mathrm{Pic}(M)_{\mathbb{Q}} = \mathrm{NS}(M)_{\mathbb{Q}}$. Then M is a Mori dream space if and only if its Cox ring $\mathrm{Cox}(M)$ is finitely generated as a $\mathbb{C}$-algebra.*

Examples of Mori dream spaces include *log Fano varieties* [BCHM10], *$\mathbb{Q}$-factorial projective toric varieties.*

For any effective $\mathbb{Q}$-divisor D on a Mori dream space M, the image of the rational map $\phi_{|mD|}$ is $\mathrm{Proj}(R(M, mD))$, which is independent of m, provided that m is sufficiently large and divisible. Denote such image by $M(D)$ and the rational map by $\phi_D : M \dashrightarrow M(D)$. Two $\mathbb{Q}$-divisor D_1 and D_2 are called *Mori equivalent* if $\phi_{D_1} = \phi_{D_2}$.

Definition C.2.3. [HK00, Definition 1.4] Let M be a projective variety such that $R(M, L)$ is finitely generated for all line bundles L and $\mathrm{Pic}(M)_{\mathbb{Q}} = \mathrm{NS}(M)_{\mathbb{Q}}$. By a *Mori chamber* of $\mathrm{NS}(M)_{\mathbb{Q}}$ we mean the closure of an equivalence class whose interior is open in $\mathrm{NS}(M)_{\mathbb{Q}}$.

Theorem C.2.4. *[HK00, Proposition 1.11] Let M be a Mori dream space. Then the following properties hold.*

(a). Mori's program can be carried out for any divisor on M. That is, the necessary contractions and flips exist, any sequence terminates, and if at some point the divisor becomes nef then at that point it becomes semi-ample.

(b). The f_i in Definition C.2.1 are the only SQMs of M. M_i and M_j in adjacent chambers are related by a flip. $\overline{\mathrm{Eff}}(M)$ is the affine hull of finitely many effective divisors. There are finitely many birational contractions $g_i : M \dashrightarrow Y_i$, with Y_i Mori dream spaces, such that

$$\overline{\mathrm{Eff}}(M) = \bigcup_i g_i^*(\mathrm{Nef}(Y_i)) \times \mathrm{ex}(g_i) \tag{C.2.1}$$

is a decomposition of $\overline{\mathrm{Eff}}(M)$ into closed convex polyhedral chambers with disjoint interiors. The cones $g_i^*(\mathrm{Nef}(Y_i)) \times \mathrm{ex}(g_i)$ are precisely the Mori chambers of $\overline{\mathrm{Eff}}(M)$. They are in one-to-one correspondence with birational contractions of M having $\mathbb{Q}$-factorial image.

(c). The chambers $f_i^*(\mathrm{Nef}(M_i))$, together with their faces, give a fan with support $\mathrm{Mov}(M)$. The cones in the fan are in one-to-one correspondence with contracting rational maps $g : M \dashrightarrow Y$, with Y normal and projective via

$$[g : M \dashrightarrow Y] \to [g^*(\mathrm{Nef}(Y) \subset \mathrm{Mov}(M)].$$

Let D be an effective divisor on M.

(d). $R(M, D)$ is finitely generated.

(e). After replacing D by a multiple, the canonical decomposition $D = A + F$ into moving and fixed part has the following properties. There is a Mori chamber containing D, so that if $g_i : M \dashrightarrow Y_i$ is the corresponding birational contraction of (b) then F has support the exceptional locus of g_i and A is the pullback of a semi-ample line bundle on Y_i.

In above (b), $\mathrm{ex}(g_i)$ is the cone spanned by the exceptional divisor of g_i, and $g_i^*(\mathrm{Nef}(Y_i)) \times \mathrm{ex}(g_i)$ denotes the join of the cones $g_i^*(\mathrm{Nef}(Y_i))$ and $\mathrm{ex}(g_i)$. That is, every element of the cone $g_i^*(\mathrm{Nef}(Y_i)) \times \mathrm{ex}(g_i)$ is uniquely written as the sum of two elements of the cones $g_i^*(\mathrm{Nef}(Y_i))$ and $\mathrm{ex}(g_i)$.

We call the decomposition (C.2.1) the *Mori chamber decomposition* of the pseudo-effective cone $\overline{\mathrm{Eff}}(M)$.

Remark C.2.5. The notion of *Mori chamber decomposition* is not restricted to Mori dream space. Here is an example, which will be used in Theorem B.2.1. Let H be a general polarization on a smooth projective K3 surface X. Let $\mathbf{v}$ be a primitive Mukai vector. Let $M = M_H(\mathbf{v})$ be the moduli space of H-Gieseker stable sheaves with Mukai vector $\mathbf{v}$. Then $\mathrm{Mov}(M)$ has a locally polyhedral chamber decomposition [BM14b] (Example 3.4.1 and Theorem 3.4.2).

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