Fourier-Mukai Transforms of Slope Semistable Sheaves on Weierstrass Elliptic Surfaces

Wanmin Liu (Uppsala Univ.)
Jason Lo (California State Univ. Northridge)

STRING MATH 2019

01-05 JULY 2019, UPPSALA, SWEDEN

Thanks to **IBS Center for Geometry and Physics** in Pohang, South Korea
Liu was supported by IBS-R003-D1
Preprint is available at wanminliu.github.io

MOTIVATION

X smooth proj var$/\mathbb{C}$

$D^b(X)$ bounded derived category of coherent sheaves

$\Phi \in \mathrm{Aut}(D^b(X))$

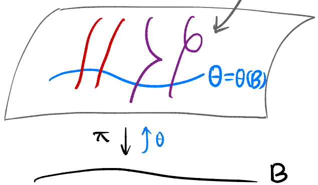
(Mumford slope, Gieseker, PT, Bridgeland etc)

$$\begin{tikzcd}
\sigma\text{- stability condition} \arrow[r, rightsquigarrow] \arrow[d, "\Phi"] & \sigma\text{-invariant} \arrow[d, dashed, "\Phi"', "\text{relation/ symmetry/ modularity ?}"] \\
\sigma' = \Phi(\sigma) \arrow[r, rightsquigarrow] & \sigma'\text{- invariant}
\end{tikzcd}$$

QUESTION

singular fibers: nodal or cusp

f elliptic



$$\begin{array}{c} X \\ \pi \big\downarrow \big\uparrow \theta \\ B \end{array}$$

elliptic fiberation with a section θ $\quad \theta^2 = -e$

over a smooth base

Φ relative Fourier–Mukai transform $\in \mathrm{Aut}(D^b(X))$ $\quad \Phi[1] \circ \hat{\Phi} = \hat{\Phi} \circ \Phi[1] = \mathrm{id}_{D^b(X)}$

$\mu_{\bar{\omega}}$ slope stability condition $\left(\mu_{\bar{\omega}}(-) = \dfrac{c_1(-) \cdot \bar{\omega}}{\mathrm{rank}(-)}, \ \mathrm{Coh}(X) \right)$

$\bar{\omega} = \underbrace{(\theta + mf)}_{\text{ample}} + \underbrace{\alpha f}_{\text{nef}}$

fixed positive number

What is a notion of stability condition for slope stability condition under $\Phi[1]$?

Key Premise: we do NOT fix Chern characters
(Otherwise, lots of work by Bruzzo, Maciocia, Yoshioka and many...)

Limit Bridgeland Stability Condition σ^{ℓ}

$$\omega = u(\theta + mf) + v f$$

along CURVE $u^2 + uv = \alpha + m - e$ ☆

$$\mathrm{Coh}(X) \xrightarrow{(\mathcal{T}_\omega,\ \mathcal{F}_\omega)}$$

$$\mathcal{B}^{\ell} \xleftarrow[\text{as } v \to \infty]{\text{limit along } \star} \mathcal{B}_\omega = \langle \mathcal{F}_\omega[1], \mathcal{T}_\omega \rangle$$

$$\sigma_\omega = \left(Z_\omega := -\int_X e^{-i\omega} ch(-),\ \mathcal{B}_\omega \right)$$

Bridgeland stab condition

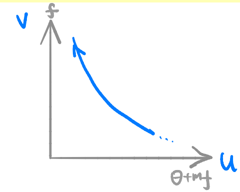
$$\boxed{\sigma^{\ell} = (Z^{\ell}, \mathcal{B}^{\ell}) := \lim_{\substack{v \to +\infty \\ \text{along } \star}} \sigma_\omega}$$

STRING MATH 2019
01-05 JULY 2019, UPPSALA, SWEDEN

Limit Bridgeland Stability Condition σ^ℓ

$$\omega = u(\Theta + mf) + v f$$

along CURVE $u^2 + uv = \alpha + m - e$ ☆



$$\begin{tikzcd}
D^b(X) & D^b(X) & \\
\mathrm{Coh}(X) \arrow[r, shift left, "\Phi[1]\ \cong"] \arrow[dr, Leftrightarrow, "\text{Thm (B)}"', "\text{Thm (A)}"] & \Phi(\mathrm{Coh}(X))[1] \arrow[l, shift left, "\hat{\Phi}"] & \\
 & \mathcal{B}^\ell \arrow[u, "/" description] & \mathcal{B}_\omega \arrow[l, "\text{limit along } \star"', "\text{as } \ v \to \infty"]
\end{tikzcd}$$

$$\mathrm{Coh}(X) \xrightarrow{(\mathcal{T}_\omega,\ \mathcal{F}_\omega)} \mathcal{B}_\omega = \langle \mathcal{F}_\omega[1], \mathcal{T}_\omega \rangle$$

$$\sigma_\omega = \left(Z_\omega := -\int_X e^{-i\omega} \mathrm{ch}(-),\ \mathcal{B}_\omega \right)$$

Bridgeland stab condition

$$\boxed{\sigma^\ell = (Z^\ell, \mathcal{B}^\ell) := \lim_{\substack{v \to +\infty \\ \text{along } \star}} \sigma_\omega}$$

Theorem (L-Lo)

$$\begin{tikzcd}
\mathrm{Coh}(X) \arrow[r, shift left, "\Phi[1]\ \simeq"] \arrow[dr, shift left, "\text{Thm (A)}"] & \Phi(\mathrm{Coh}(X))[1] \arrow[l, shift left, "\hat{\Phi}"] \\
 & \mathcal{B}^{l} \arrow[ul, shift left, "\text{Thm (B)}"] \arrow[u, "/"]
\end{tikzcd}$$

X Weierstrass elliptic surface

(A) $(\mu_{\bar{\omega}}\text{-stability}, \mathrm{Coh}(X)) \xrightarrow{\Phi[1]} (Z^{l}\text{-stability}, \mathcal{B}^{l})$

E - $\mu_{\bar{\omega}}$ stable $\qquad$ $F := \Phi(E)[1]$ Z^{l}-stable

$\downarrow \qquad\qquad \downarrow$

(B) $(\mu_{\bar{\omega}}\text{-semistability}, \mathrm{Coh}(X)) \xleftarrow{\hat{\Phi}} (Z^{l}\text{-semistability}, \mathcal{B}^{l})$

small modification

$\hat{\Phi}(F')$ - $\mu_{\bar{\omega}}$ - s.s. $\qquad$ $\exists \; 0 \to F' \to F \to F'' \to 0$ in $\mathcal{B}^{l}$

$\hat{\Phi}(F'')$ torsion sheaf