

Inequality and the Zero Lower Bound

Jesús Fernández-Villaverde
University of Pennsylvania
NBER, CEPR

Joël Marbet
CEMFI

Galo Nuño
Banco de España

Omar Rachedi
ESADE Business School
Universitat Ramon Llull

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Abstract

This paper studies how household inequality shapes the effects of the zero lower bound (ZLB) on aggregate dynamics. To do so, we consider a heterogeneous agent New Keynesian model with an occasionally binding ZLB, and solve for its fully non-linear stochastic equilibrium using a novel neural-network algorithm. In this setting, changes in the monetary policy stance influence households' precautionary savings by altering the frequency of ZLB events. As a result, the model features a long-run Fisher equation, through which an increase in the central bank's inflation target raises the level of real interest rates. While the magnitude of monetary super-neutrality is limited on average, long-run real rates may vary substantially with the inflation target at high levels of wealth inequality.

Keywords: Heterogeneous Agents, Neural Networks, Non-linear Dynamics.

JEL Classification Codes: D31, E12, E21, E31, E43, E52, E58.

1 Introduction

Recent research has highlighted the key role of income and wealth inequality in the transmission mechanism of monetary policy through the lens of Heterogeneous Agent New Keynesian (HANK) models (McKay, Nakamura and Steinsson, 2016; Kaplan, Moll and Violante, 2018; Auclert, 2019). However, this body of work tends to either abstract from the possibility that policy rates are constrained by the zero lower bound (ZLB), or incorporates the ZLB while imposing perfect foresight (McKay, Nakamura and Steinsson, 2016; Guerrieri and Lorenzoni, 2017). This stands in contrast to the literature on representative-agent New Keynesian (RANK) models.¹ In particular, Adam and Billi (2007), Nakov (2008), Hills, Nakata and Schmidt (2019), and Bianchi, Melosi and Rottner (2020) show that, when agents anticipate future ZLB changes, changes in the stance of monetary policy open the door to a deflationary bias and eventually affect the long-run level of interest rates. Thus, long-run interest rates are not uniquely pinned down by structural parameters unrelated to (monetary and fiscal) policy, as in standard macroeconomic models. In this paper, we revisit the role of the ZLB on shaping aggregate dynamics in a setting with heterogeneous agents, and uncover to what extent household inequality may rationalize an empirically relevant long-run Fisher equation.

Specifically, we extend the workhorse RANK model in two ways. First, we consider household heterogeneity due to uninsurable idiosyncratic labor income risk, in the spirit of Bewley (1980), Huggett (1993), and Aiyagari (1994). Second, we explicitly account for the risk of hitting the ZLB in households' expectations, as in Fernández-Villaverde et al. (2015). While the structure of the economy is kept as simple as possible to transparently uncover how the implications of the ZLB constraint vary with inequality, our framework generalizes the HANK model by incorporating the non-linear dynamics implied by the existence of an occasionally binding ZLB. To solve the model, we propose a novel neural-network algorithm that allows for the computation of the stochastic equilibrium of a HANK economy. Rather than using standard computation methods for heterogeneous-agent models, based on linearization around the deterministic steady state,² we build on the global approach of Fernández-Villaverde, Hurtado and Nuño (2020). In this way, we can solve a non-linear heterogeneous-agent model via the use of neural networks. As long as neural networks can approximate the non-linear laws of motion of the states of the economy generated by the presence of the ZLB, we can derive the fully non-linear solution of the model.

¹For the studies that evaluate the implications of the ZLB by solving for the stochastic dynamics of RANK economies, see Christiano, Eichenbaum and Rebelo (2011), Coibion, Gorodnichenko and Wieland (2012), and Andrade et al. (2019)).

²See for instance Krusell and Smith (1998), Reiter (2009), Ahn et al. (2018), and Auclert et al. (2021).

Our economy features a continuum of ex-ante homogeneous households, which are ex-post heterogeneous due to an uninsurable idiosyncratic labor earning risk. Households can smooth the effect of labor earning risk on their consumption via borrowing. However, their credit capacity is limited by a borrowing constraint. On the production side, there is a continuum of intermediate-good firms that produce using labor and set prices subject to Rotemberg (1982) adjustment costs. The model also features a monetary authority that sets the nominal interest rate with a Taylor rule, subject to the ZLB constraint. Finally, we consider preference shocks as the source of aggregate uncertainty following the extensive literature that shows how changes in the degree of households’ patience can be a main driver of a ZLB event (Christiano, Eichenbaum and Rebelo, 2011).³

The quantitative analysis is disciplined by the one-asset calibration of McKay, Nakamura and Steinsson (2016) and Kaplan, Moll and Violante (2018). Specifically, our economy is consistent with the dispersion of labor earnings and liquid wealth across households, while generating a realistic value for the average marginal propensity to consume. We set the volatility of the aggregate preference shocks such as the stochastic steady state (SSS) – that is, the steady state of the non-linear model when the shock realizations are zero – of our baseline economy with a 2% inflation target features a 10% ZLB frequency, as in the data (Coibion et al., 2016).

We find that the level of the real interest rate in the SSS of our economy is lower than in the SSS of an equivalently parameterized RANK economy. The rationale is twofold. First, household idiosyncratic risk reduces real rates in the deterministic steady state (DSS). Although this result traces back to Aiyagari (1994), our setting grants it a novel perspective, as precautionary savings reduce the room of manoeuvre for the central bank’s policy rate, and consequently alter the ZLB frequency as well as the behavior of the macroeconomic variables in the SSS. Importantly, this result does not emerge in the standard HANK literature, in which the drop in the real rate due to precautionary savings is immaterial for aggregate dynamics because of the lack of the ZLB. Second, the interaction of aggregate uncertainty with wealth heterogeneity increases precautionary savings at the SSS, thus exerting a downward force in the real interest rate vis-à-vis the RANK economy. This effect builds on the premise that ZLB recessions hit disproportionately more labor income. Since the total earnings of wealth-poor agents mainly relies on wages, they experience a relatively larger drop in income, which is then coupled with their relative higher marginal propensity to consume. In other words, ZLB events affect disproportionately more wealth-poor agents. Consequently, the possibility of the occurrence of large recessions in which the policy rate is

³The preference shocks are a reduced-form for any variation in uncertainty, fiscal policy, or credit market tightness that could alter households’ risk appetite.

constrained leads households – and especially those at the bottom of the wealth distribution – to increase their buffer of precautionary savings.

These two channels imply that, as long as the monetary policy strategy alters the frequency of ZLB events, it would also modify households’ decisions about their optimal buffer of precautionary savings, and ultimately affect the level of the real interest rate in the SSS. Thus, our model generates a long-run Fisher equation in which the long-run real rate depends positively on the central bank’s inflation target. That is, if we denote the steady-state nominal interest rate, real rate, inflation rate, and inflation target by i , r , π , and $\bar{\pi}$, respectively, then $i(\bar{\pi}) = r(\bar{\pi}) + \pi(\bar{\pi})$, such that $dr/d\bar{\pi} > 0$. In our baseline calibration, household heterogeneity does not substantially influence the long-run Fisher equation. Specifically, we consider an exercise in which we move the inflation target from 4% to 1.7%, which correspond to the average inflation rates between 1980 and 1999 and from the year 2000 on, respectively. In this case, our economy implies a drop in the real rate which exceeds that of the RANK economy by just 4 bps. From this perspective, the model is consistent with the large empirical literature that has found a limited support for monetary super-neutrality (Lucas, 1980; Bullard and Keating, 1995; Evans and Lewis, 1995; King and Watson, 1997; Bernanke and Mihov, 1998; Ahmed and Rogers, 2000).

However, the long-run Fisher equation becomes empirically relevant at high levels of inequality. To uncover this result, we jointly consider a reduction in the inflation target from 4% to 1.7% with a rise in wealth Gini by three percentage points, which is the change in wealth inequality observed in the U.S. in the early 2000s. In this case, our model implies a drop in the real rate which exceeds that predicted by the RANK economy by 24 bps, which accounts for roughly 10% of the decline in real rates over the last decades. Thus, the monetary super-neutrality of our model is state dependent, such that the long-run Fisher equation becomes much steeper at high degrees of wealth inequality. In addition, the interaction of rising inequality and frequent ZLB events rationalizes the evidence in Hillenbrand (2021) and Bianchi, Lettau and Ludvigson (2022) showing that monetary policy is responsible the secular real rate decline. In this way, we provide a novel explanation to the reduction in real rates that complements additional explanations such as population aging (Carvalho, Ferrero and Nechio, 2016; Aksoy et al., 2019), the surge in convenience yields (Krishnamurthy and Vissing-Jorgensen, 2012), and differences in households’ marginal propensity to save out of permanent income (Mian, Straub and Sufi, 2021).

Our results on the positive degree of monetary super-neutrality at low levels of inflation complements the implications of the Generalized New Keynesian model (Ascari, 2004; Ascari and Sbordone, 2014), in which higher levels of the inflation rate reduce equilibrium output. However, this setting does not feature a long-run Fisher equation. Our paper also connects to

Auclert and Rognlie (2020), who analyze the link between inequality and aggregate demand. They find that an increase in income inequality in the deterministic steady state can lead to a substantial decline of consumption, output, and the real rate, especially when the monetary authority is always constrained by the ZLB.⁴ Instead, our paper looks at how inequality alters the *ex-ante* incentives to accumulate precautionary savings when households face an occasionally binding ZLB, a mechanism that affects the levels of consumption, output, and the real rate in the stochastic steady state.

This paper relates to a rising strand of the literature which studies the full stochastic dynamics of HANK economies with aggregate shocks. For instance, in independent work, Gorodnichenko et al. (2021) evaluates how stochastic volatility in aggregate productivity affects household consumption-saving decisions. Schaab (2020) introduces an alternative method based on the projection over basis functions to uncover how the interaction of macro and micro volatility can lead the economy into a downward aggregate demand spiral towards a liquidity trap. Finally, Kase, Melosi and Rottner (2022) introduces the use of neural networks to estimate HANK economies subject to the ZLB constraint. From the methodological contribution, our paper builds on Fernández-Villaverde, Hurtado and Nuño, who pioneered the use of neural networks to globally solve heterogeneous-agent models. Our approach relates to Maliar, Maliar and Winant (2021) and Azinovic, Gaegauf and Scheidegger (2022), who also employ deep learning to solve globally heterogeneous-agent models with aggregate shocks.

2 Model

2.1 Environment

The economy is populated by a unit measure of ex-ante homogeneous households, which are ex-post heterogeneous due to the realizations of a persistent uninsurable labor earning risk. The idiosyncratic shock follows a Markov chain process. Households' utility function over consumption and leisure is subject to an aggregate preference shock, which follows an AR(1) process. Households finance consumption expenditures with labor income as well as firms' profits, and are allowed to borrow up to an exogenous limit.

The production side of the economy consists of two levels. There is a continuum of intermediate good producers, which produce different varieties of the intermediate good using labor, subject to a Rotemberg (1982) price-setting cost. Then, the different varieties of the intermediate goods are bundled into the final consumption good by the final good

⁴In a similar spirit, McKay and Reis (2016) show that the presence of the ZLB matters for the overall effects of automatic stabilizers.

producer.

Finally, the economy features a monetary authority that sets the nominal interest rate following a standard Taylor rule, subject to the ZLB constraint, as well as a fiscal authority, which levies progressive labor income taxes on the households to finance a fixed amount of outstanding public debt.

2.2 Households

There is a unit measure of ex-ante identical households indexed by $i \in [0, 1]$. Households face idiosyncratic and aggregate risk. The idiosyncratic labor earning shock $s_{i,t} \in \{s_m\}_{m=1}^M$ determines the efficiency unit of hours supplied by each household, and follows a Markov chain with transition matrix Ω . We normalize the process such that the average realization of the idiosyncratic shock is $\int s_{it} di = 1$. The aggregate shock ξ_t is a preference shifter, and follows an AR(1) process in logs

$$\log \xi_t = \rho_\xi \log \xi_{t-1} + \zeta_t, \quad (1)$$

where $\rho_\xi \in (0, 1)$ and ζ_t is normally distributed with mean 0 and variance σ_ξ . The use of the demand shocks follow the work of Krugman, Dominquez and Rogoff (1998), Eggertsson et al. (2003), and Eggertsson and Krugman (2012), which show that shifts in households' preferences – a reduced form for any change that affect households' leveraging capacity – is a powerful driver of ZLB events:

Households choose consumption $c_{i,t}$, bonds $b_{i,t}$ and labor services $h_{i,t}$ to maximize life-time expected discounted utility

$$\max_{\{c_{i,t}, b_{i,t}, h_{i,t}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \xi_t u(c_{i,t}, h_{i,t}) \quad (2)$$

$$\text{s.t. } c_{i,t} + b_{i,t} = (1 - \tau) w_t s_{i,t} h_{i,t} + \Pi_t s_{i,t} + \frac{R_{t-1}}{\pi_t} b_{i,t-1}, \quad (3)$$

$$b_{i,t} \geq \underline{b}. \quad (4)$$

where β denotes the time-discount parameter. Households' optimization problem is subject to the budget constraint in Equation (3) and the borrowing constraint in Equation (4). The budget constraint posits that households finance consumption expenditures with firm profits Π_t , which are rebated to households depending on their idiosyncratic labor productivity, and labor earnings, $w_t s_{i,t} h_{i,t}$, where w_t denotes the real wage. Labor earnings are subject to a tax rate τ . Households also trade one-period non-contingent bonds, which yield the gross nominal return R_{t-1} . We then refer to the gross real return on bonds as $r_t = \frac{R_{t-1}}{\pi_t}$, which divides the gross nominal return by gross inflation. The borrowing constraint posits that

households bond position is limited by the exogenous limit \underline{b} .

Households have GHH preferences as in Greenwood, Hercowitz and Huffman (1988), such that

$$u(c_{i,t}, h_{i,t}) = \frac{1}{1-\sigma} \left(c_{i,t} - \chi \frac{h_{i,t}^{1+\nu}}{1+\nu} \right)^{1-\sigma}, \quad (5)$$

where σ denotes the risk-aversion parameter, ν is the inverse of the Frisch elasticity, and χ is a leisure shifter.

2.3 Firms

2.3.1 Final Good Firm

There is a final good that is produced by a representative final good firm and sold in a perfectly competitive final good market. The final good firm produces the final good Y_t by bundling together a continuum of intermediate inputs, $y_{j,t}$, indexed by $j \in [0, 1]$, by means of a CES production function

$$Y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}, \quad (6)$$

where ε is the elasticity of substitution between the different intermediate-input varieties. This CES aggregator yields the following optimal iso-elastic demand for intermediate good j

$$y_{j,t} = \left(\frac{p_{j,t}}{P_t} \right)^{-\varepsilon} Y_t, \quad (7)$$

where $p_{j,t}$ is the price of variety j , and P_t is the aggregate price level. These two price indexes are related together through the following equation,

$$P_t = \left(\int_0^1 p_{j,t}^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}. \quad (8)$$

2.3.2 Intermediate Good Firms

Each intermediate good firm j produces its variety using labor services $l_{j,t}$ according to a linear production function

$$y_{j,t} = l_{j,t}. \quad (9)$$

which implies that marginal costs $m_{j,t}$ equal wages w_t

$$m_{j,t} = w_t, \quad (10)$$

so that marginal costs – and hired labor – are equalized across producers, that is, $m_{j,t} = m_t$ and $l_{j,t} = l_t$, for all $j \in [0, 1]$.

Intermediate good firms cannot freely adjust their prices, but rather face price adjustment costs as in Rotemberg (1982). This feature introduces nominal rigidities in the model. We follow Bayer et al. (2019) in the specification of the adjustment costs, with the only difference that we extend their settings to the case of a trend inflation possibly different from zero. In this case, the adjustment costs, $\Theta_{j,t}$, take the following functional form

$$\Theta_{j,t} \equiv \Theta \left(\frac{p_{j,t}}{p_{j,t-1}} \right) = \frac{\theta}{2} \left[\log \left(\frac{p_{j,t}}{p_{j,t-1} \times \tilde{\pi}} \right) \right]^2 Y_t, \quad (11)$$

where $\tilde{\pi}$ denotes the trend inflation which is targeted by the central bank, and θ crucially captures the degree of price stickiness. When $\theta = 0$, the economy does not feature nominal rigidities, and therefore collapses to a standard neoclassical heterogeneous-agent model with imperfect competition on the supply side.

Given the price adjustment costs, the problem of intermediate good producers is to choose a sequence of prices $\{p_{j,t}\}_{t \geq 0}$ to maximize the expected discounted stream of profits net of the adjustment costs,

$$\mathbb{E}_t \sum_{k=t}^{\infty} \beta^k \left[\Pi_k(p_{j,k}) - \Theta \left(\frac{p_{j,k}}{p_{j,k-1}} \right) \right], \quad (12)$$

where the profits of the intermediate good firm, $\Pi_k(p_{j,k})$, equal

$$\Pi_k(p_{j,k}) = \left(\frac{p_{j,k}}{P_k} - m_k \right) \left(\frac{p_{j,k}}{P_k} \right)^{-\varepsilon} Y_k. \quad (13)$$

In the pricing protocol, we follow Hagedorn, Manovskii and Mitman (2019) by assuming that the price adjustment costs are virtual. Basically, the adjustment costs do affect firms' decision of setting optimal prices, but eventually do not result in any transfer of real resources. Accordingly, the profits of each firm, $\Pi_t(p_{j,t})$, describe the entire resources that are rebated back to the households. In this procedure, we further assume that profits are rebated to each household according to its idiosyncratic productivity level, such that we are consistent with the empirical observation that earnings-rich households tend to receive a disproportionately larger share of firm profits.

Finally, the overall solution of the problem of the intermediate good firms yields the following New Keynesian Philips curve

$$\log \left(\frac{\pi_t}{\tilde{\pi}} \right) = \beta E_t \left[\log \left(\frac{\pi_{t+1}}{\tilde{\pi}} \right) \frac{Y_{t+1}}{Y_t} \right] + \frac{\varepsilon}{\theta} \left(m_t - \frac{\varepsilon - 1}{\varepsilon} \right), \quad (14)$$

where $\pi_t = \frac{P_t}{P_{t-1}}$ is gross inflation rate.

2.4 Government

The government consists of a monetary authority and a fiscal authority. The monetary authority consists of a central bank which sets the gross nominal interest rate R_t according to a standard Taylor rule

$$R_t = \max \left\{ 1, \tilde{R} \left(\frac{\pi_t}{\tilde{\pi}} \right)^{\phi_\pi} \left(\frac{Y_t}{\tilde{Y}} \right)^{\phi_y} \right\},$$

\tilde{R} is the gross nominal rate in the deterministic steady state, and \tilde{Y} denotes the level of output in the deterministic steady state.

This specification of the Taylor rule implies two important considerations. First, the monetary authority faces a ZLB constraint, such that the gross nominal interest rate cannot go below one. Second, when the central bank is not constrained by the ZLB, then it sets the nominal interest rate by reacting to changes in the inflation rate from its target – the parameter ϕ_π determines the strength with which this happens – and by reacting to changes in output from its steady state level – the parameter ϕ_y pins down the strength of this second channel.

On the other hand, the fiscal authority raises labor earning taxes on the households to finance a fixed amount of outstanding debt \tilde{B} , such that the government budget constraint equals

$$\int_0^1 \tau w_t s_{i,t} h_{i,t} di = (r_t - 1) \tilde{B}. \quad (15)$$

We consider a fixed amount of outstanding debt so that we can calibrate our economy to be consistent with the amount of liquid assets of the U.S. economy, in line with the calibration strategy of McKay, Nakamura and Steinsson (2016) and the one-asset economy of Kaplan, Moll and Violante (2018). Given the level of debt and the equilibrium interest rate, the tax rate τ is set to clear the budget constraint.

2.5 Market Clearing Conditions

A competitive equilibrium for this economy implies the following market clearing conditions. First, the labor market clears, so that the overall efficiency units of hours provided by households equal the labor services demanded by the intermediate good firms, that is,

$$\int_0^1 l_{j,t} dj = \int_0^1 s_{i,t} h_{i,t} di. \quad (16)$$

Second, the bond market clears, so that the overall bond positions of households equal the outstanding government bonds issued by the fiscal authority, that is,

$$\tilde{B} = \int_0^1 b_{i,t} di. \quad (17)$$

Finally, the aggregate resource constraint posits that total output equals the sums of the value added of the intermediate good firms, and also equals the overall consumption of the households, that is,

$$Y_t = \int_0^1 l_{j,t} dj = \int_0^1 c_{it} di. \quad (18)$$

3 Calibration

We calibrate the model to the U.S. economy by setting one time period of the model to correspond to a quarter. Table 1 shows the current parametrization of the model and the chosen targets. First, we set the gross inflation target of the monetary authority to $\tilde{\pi} = \exp(0.02/4)$, so that the annual inflation target is 2% in the baseline economy. As far as the Taylor rule parameters are concerned, we set the sensitivity of the nominal rate to output deviations from the steady state to $\phi_y = 0.1$, and that to changes in inflation to $\phi_\pi = 2.5$. Finally, to discipline the level of the real rate in the deterministic steady state, we set the time discount factor to $\beta = 0.9954$. This choice implies that the real interest rate in the deterministic steady state equals 1.5%. Our choice binds the value of the real interest in the deterministic steady state to a value which is in the ballpark of the estimates provided by the literature (Del Negro et al., 2017; Fiorentini et al., 2018), and also consistent with the November 2018 FOMC’s Summary of Economic Projections. However, in the quantitative results we will show that different values of the inflation target imply a large variation in the level of real interest rates in the stochastic steady state. This happens even if all the economies share the same value of real interest rates in the deterministic steady state.

Next, we calibrate the demand shock process, which is the only source of aggregate risk in our model. Since we want to understand how household heterogeneity and changes in the monetary policy stance interact alter households’ decision by modifying the risk of hitting the ZLB, we require our model to be consistent with the frequency of ZLB episodes observed in the U.S. economy. To do so, we first set the persistence of the autoregressive process to $\rho_\xi = 0.6$, in line with the parametrization of Bianchi, Melosi and Rottner (2020), and then set the standard deviation to $\omega_\xi = 0.0113$, so that – under a 2% inflation target – the model reproduces a 10% ZLB frequency, in line with the frequency observed in the U.S. in the post-war period (Coibion et al., 2016).

As far as the calibration of the idiosyncratic risk is concerned, we want our economy

Table 1: Baseline Parametrization

Parameter		Value	Target/Source
Panel A. Aggregate Risk			
ρ_ξ	AR coefficient of process for ξ	0.6	Bianchi, Melosi and Rottner (2020)
ω_ξ	Standard deviation of ξ shock	0.0113	10% ZLB frequency
Panel B. Idiosyncratic Risk			
ρ_s	AR coefficient of process for s_t	0.94	10% Average MPC
ω_s	Standard deviation of s_t shock	0.075	30% Borrowers
\underline{b}	Borrowing limit	-0.58	Two-month average labor income
Panel C. Preferences			
β	Discount factor	0.9954	1.5% real interest rate in the DSS
σ	Risk aversion	1	Standard value
$1/\nu$	Frisch elasticity of labor supply	1	Standard value
χ	Disutility of labor	0.8707	Labor supply equals 1 in the DSS
Panel D. Production			
ε	Demand elasticity	7.67	15% price markup
θ	Rotemberg price adjustment cost	117	12-month duration of prices
Panel E. Monetary Authority			
$\tilde{\pi}$	Inflation Target	$\exp(0.02/4)$	2% Annual inflation target
ϕ_π	Taylor rule coefficient on inflation	2.5	Standard value
ϕ_y	Taylor rule coefficient on deviations from steady state output	0.1	Standard value
Panel F. Fiscal Authority			
B	Government outstanding bonds	1	Total liquid assets = 25% annual GDP

to be consistent with the share of borrowers and savers observed in the data, and with the empirical estimates of the marginal propensity to consume (MPC). To do so, we first set the borrowing limit to $\underline{b} = -0.58$, so that it equals two monthly average wages. Given this tightness of the borrowing constraint, we calibrate the labor earning process so that the model reproduces a share of borrowers of 30% (Kaplan, Violante and Weidner, 2014), and an average MPC of around 10%, which is at the lower end of the estimates provided by the literature (Johnson, Parker and Souleles, 2006; Parker et al., 2013; Broda and Parker, 2014). These values are obtained by setting the parameters of the AR(1) process governing the changes in households' efficiency unit of hours to $\rho_s = 0.94$ and $\omega_s = 0.075$. We then convert the AR(1) process into a three point Markov Chain using the Rouwenhorst (1982) method.

This parametrization leads to a Gini index of wealth of 0.81, which replicates exactly the value of the Gini index that Kuhn and Ríos-Rull (2016) find for the U.S. economy in the early 2000s.

After having disciplined the amount of aggregate and idiosyncratic uncertainty, we calibrate the parameters of the households as follows. We start by fixing the risk aversion to the standard value of $\sigma = 1$, and the Frisch elasticity of labor supply to $1/\nu = 1$. This value is slightly higher than that proposed by Chetty et al. (2013) based on the elasticity of labor supply estimated at the micro level, but it is nonetheless at the lower end of the values typically considered by the literature on the transmission of monetary policy. Finally, we normalize the disutility of labor to $\chi = 0.8707$ such that the aggregate value of the efficiency units of hours equals one in the deterministic steady state.

On the production side, we set the demand elasticity across different intermediate-input varieties to $\epsilon = 7.67$, such that the price markup is 15%, in line with the values commonly used in the New Keynesian literature (e.g., Christiano, Eichenbaum and Evans, 2005). The price adjustment parameter is set to 117 using the equivalence with respect to the Calvo-price protocol and target a price duration in line with the 12 months usually used in the literature.

Regarding the fiscal authority, the government finances a fixed amount of outstanding bonds which amount to 25% of annual GDP, in line with the estimate of liquid wealth in the U.S. economy derived by Kaplan, Moll and Violante (2018).

4 Algorithm

The computation of heterogeneous agent is complicated by the fact that an equilibrium requires the households to keep track of how the distribution of bonds evolves over time, so that it allows them to be able to properly form expectations. However, this feature makes the solution computationally intractable as one needs to keep track of an infinite-dimensional object. Thus, this class of model is often solved by assuming bounded rationality, as in Krusell and Smith (1998). Basically, the solution algorithm imposes that households need to keep track of just a moment of the bond distribution, and this moment tends to be the mean. Then, the Krusell and Smith (1998) hinges on linear law of motion approach, in which the individual policy function first solves for the updated individual policy function, and then updates the perceived law of motion derived within a stochastic simulation. The algorithm then repeats these steps until the convergence of the perceived law of motion.

However, our economy is inherently non-linear, and therefore the linear approach of Krusell and Smith (1998) would not capture the full dynamics of the economy. An important innovation of this paper is to introduce the solution of a non-linear economy within the

literature of HANK models. This allows us to evaluate the stochastic steady state in an economy that features price rigidities, heterogeneous agents, and a zero lower bound. In doing so, we follow the approach of Fernández-Villaverde, Hurtado and Nuño (2020), showing that neural networks can fully approximate in a non-parametric way the non-linearity of the law of motion of a heterogeneous agent economy. We report a thorough description of the algorithm in Appendix A.

Figure 1: Non-Linearity due to the ZLB.

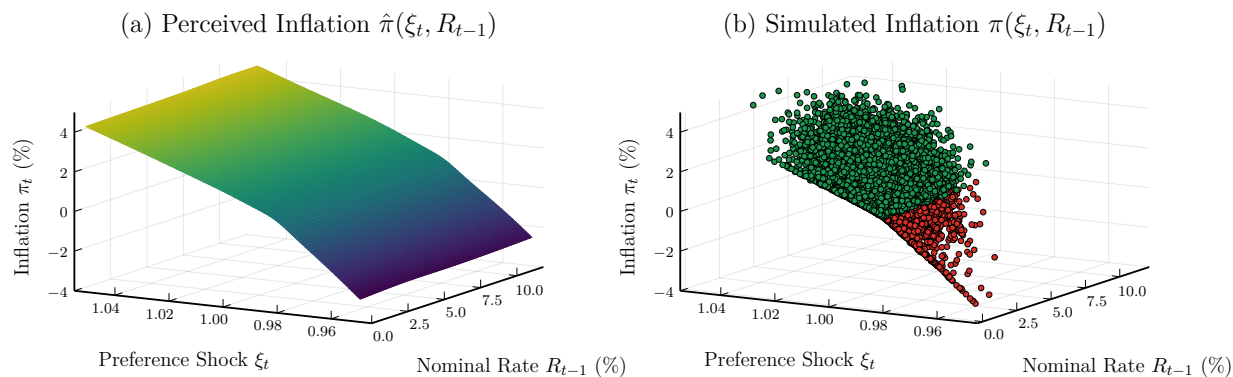
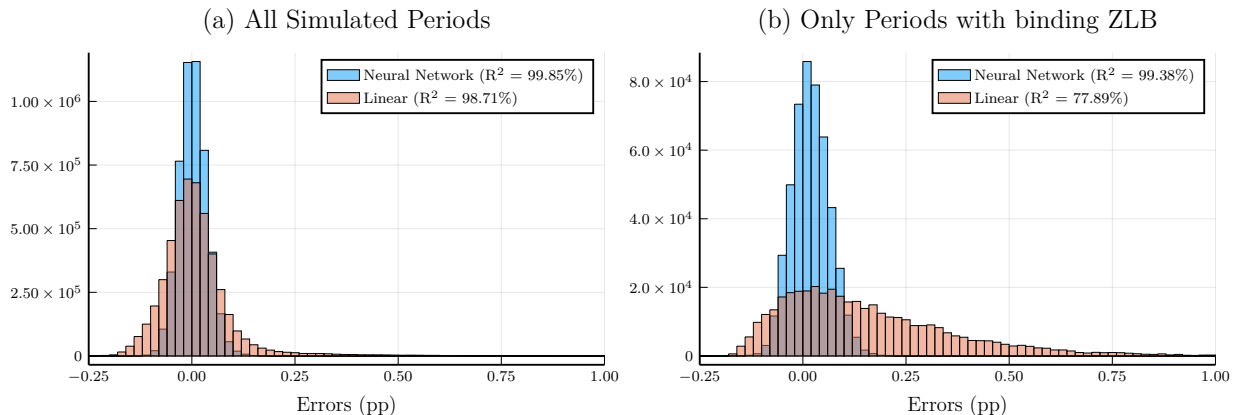


Figure 1 shows how our algorithm uncovers a marked non-linearity in both in both the perceived and simulated inflation rate, a non-linearity which is generated by the presence of the zero lower bound, which holds for sufficiently low levels of the realizations of the demand shock process ξ_t .

To further corroborate the relevance of our approach, Figure 2 compares the nowcast errors on the dynamics of inflation generated by our neural network approach, with those implied by the naive application of the Krusell and Smith (1998) method, in which we predict the inflation rate as a log-linear function of the state variables. Panel (a) reports the errors for all simulated periods, and shows that the neural network approach increases the fit of households' expectations by increasing the R^2 from 98.71% to 99.85%. This improvement, is concentrated in the left-hand part of the distribution of the errors, that is, cases in which inflation tend to be lower than expected due to the deflationary spirals generated amidst a ZLB event. Indeed, our approach is able to capture properly the non-linear dynamics at the ZLB.

Panel (b) shows that the linear Krusell and Smith (1998) provides a very poor fit for inflation dynamics when the ZLB constraint binds, with an R^2 of 77.89%, whereas the neural network algorithm generates an R^2 of 99.38%. Hence, neural networks allow us to properly characterize the non-linear dynamics of the stochastic steady state at the ZLB, a feature that would not be present if we were to rely on standard computational algorithms.

Figure 2: Nowcast Errors for Inflation.



5 Results

We use the calibrated model to evaluate to what extent the presence of the ZLB constraint on the nominal rates set by the monetary authority alters the behavior of household-level and aggregate-level variables in a HANK economy, and uncover how the differences in the dynamics between our ZLB-HANK and the standard HANK economy depend crucially on both the level of the inflation target and the amount of the wealth inequality.

We start by comparing the responses of aggregate variables and households' optimal decision to the realizations of the aggregate demand shock in a HANK economy and in our ZLB-HANK model. We then illustrate how the presence of the ZLB constraint alters the stochastic steady state of the economy. Finally, we consider a set of exercises in which we either vary solely the inflation target of the monetary authority, or we change both the inflation target and the amount of wealth inequality.

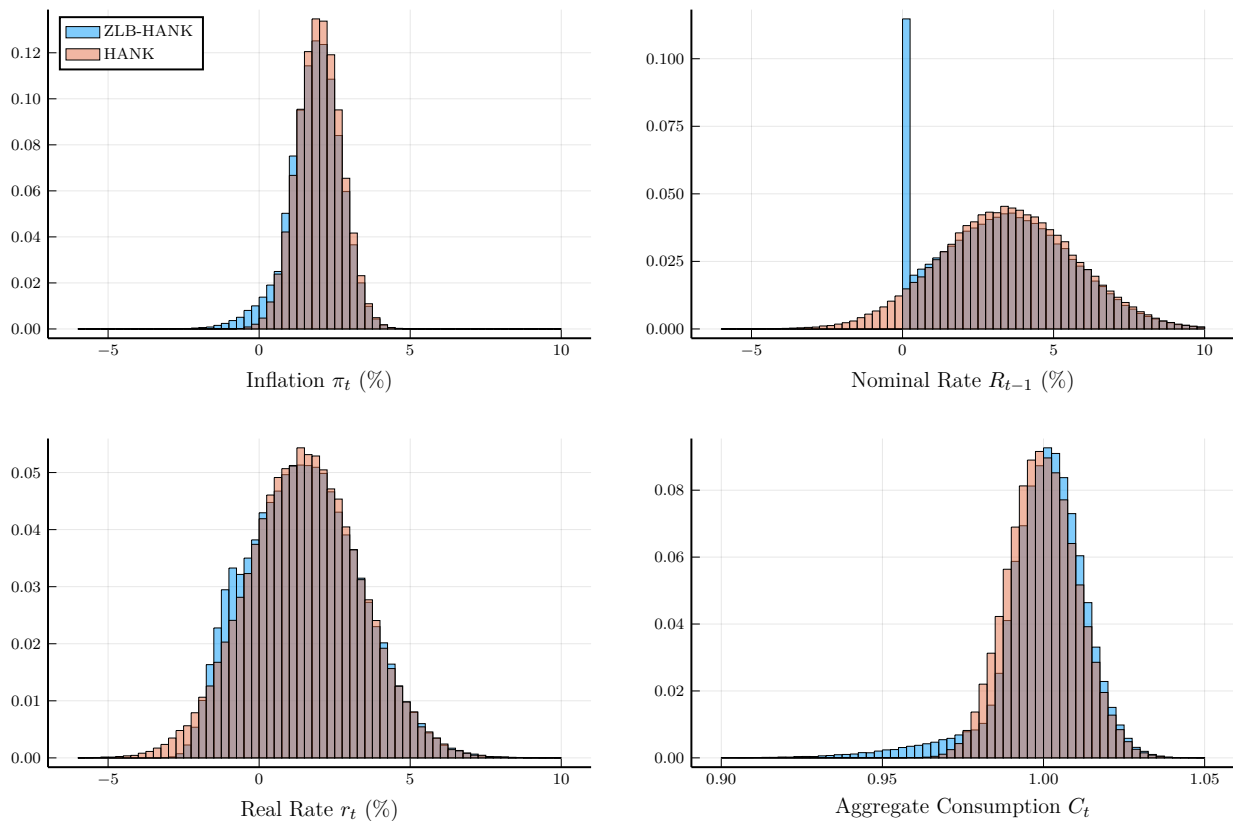
5.1 The Macro of the ZLB: Aggregate Impulse-Response Functions

Our paper is the first to study the stochastic dynamics of a heterogeneous agent model by explicitly taking into account the non-linearity generated by the ZLB constraint. A natural question is that to understand the qualitative and quantitative implication of the non-linearity on the dynamics at the stochastic steady state.

The relevance of the non-linearity generated by the presence of the ZLB constraint can be appreciated by comparing the ergodic distribution of the key variables of our ZLB-HANK model with the implications of a counterfactual version of the baseline economy that abstracts from the the ZLB, that is, the standard HANK model.

Figure 3 reports the ergodic distribution of inflation, the nominal interest rate, the real

Figure 3: Ergodic Distributions.

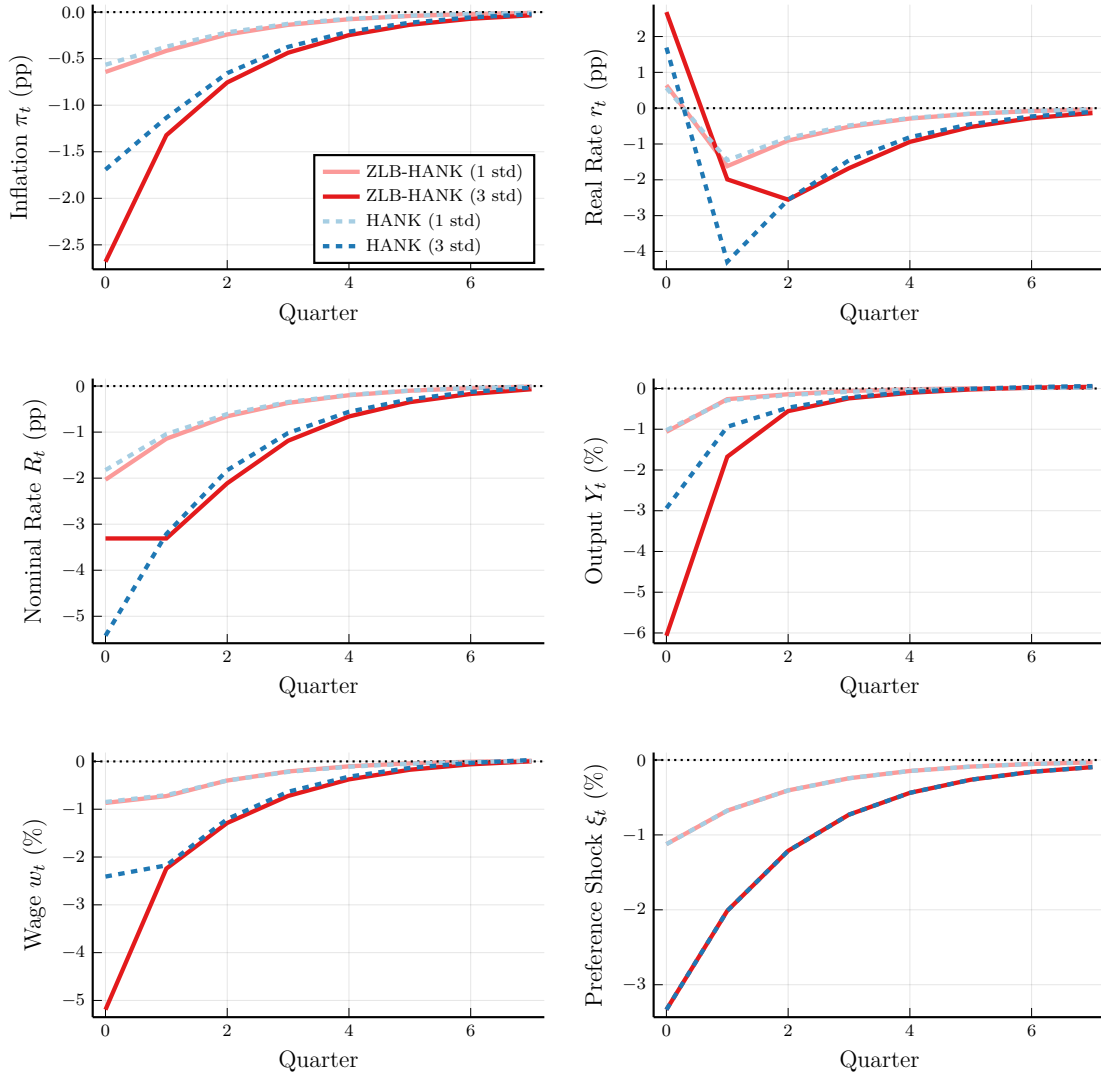


interest rate, and aggregate consumption in both the baseline ZLB-HANK and the HANK economy. The graph shows how the presence of the ZLB skews the dynamics of the model to the left: it corresponds to cases in which the nominal interest rate is constrained by the ZLB, and the economy experiences a sharp drop in aggregate consumption amidst a deflationary spiral. All these dynamics are absent in the standard HANK model.

The effect of the non-linearity of the ZLB constraint on the dynamics of inflation and output can be also evaluated by looking at the impulse-response function to small and large demand shocks (i.e., a one standard deviation shock and a three standard deviation shock, respectively) in both the ZLB-HANK and the HANK economy, which we report in Figure 4.

The plot shows that the impact response of the model to small demand shocks coincide both in the model with and without ZLB, as the shock is too small to generate a sufficiently large drop in the nominal interest rate that makes the ZLB binding. Instead, the responses to large demand shocks differ starkly depending on the presence or not of the ZLB. In the baseline model with the ZLB, a large shock brings the nominal interest rate down to zero, and it cannot go down further. This situation is accompanied by a much larger drop in both inflation and output, as the monetary authority cannot reduce the interest rate to be as

Figure 4: Impulse Response Functions.



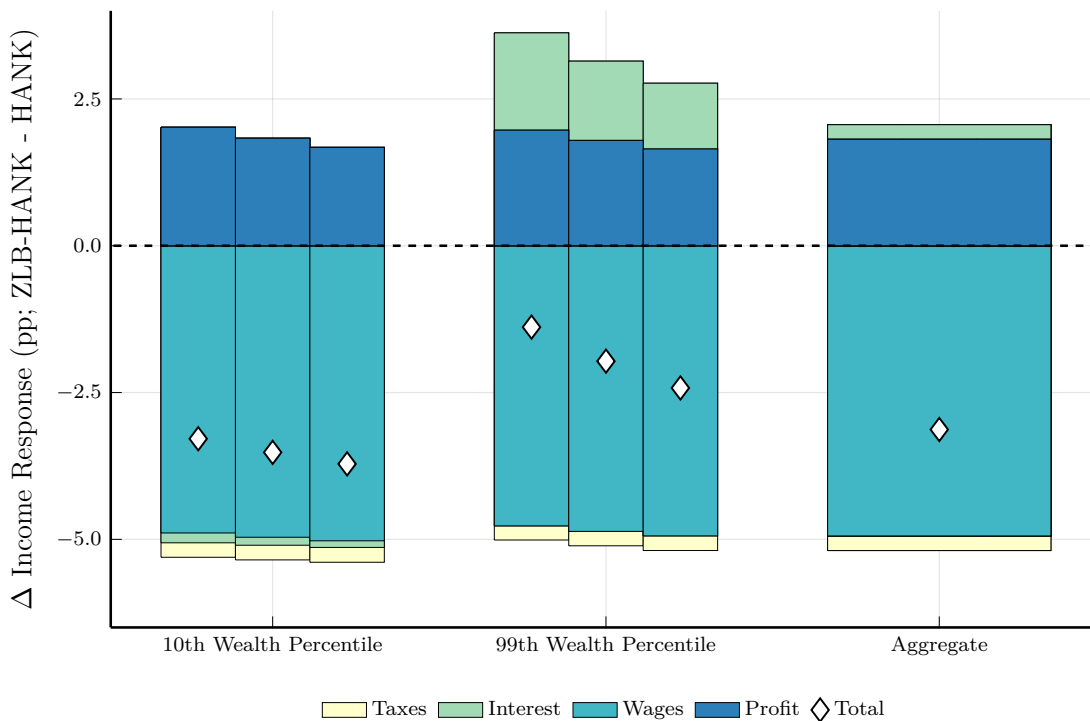
accommodative as it should be to minimize the negative effects of such a large shock. Figure 4 then highlights the important asymmetries generated by the ZLB: output and inflation drop much following big negative shocks when the constraint binds. Accordingly, ZLB events are characterized by deflation and substantial consumption losses for all households.

5.2 The Micro of the ZLB: Households' Impulse-Response Functions

The previous sector has highlighted how the presence of the ZLB constraint alters substantially the macroeconomic dynamics in response to (large realizations of) the demand shocks. However, the ZLB also alters the distributional consequences over business cycle fluctua-

tions. To establish this result, we report in Figure 5 the differential impact response of total income to a negative demand shock by splitting total income across all its different components. Specifically, we compute the difference in the response of total income in the ZLB-HANK and the HANK economy (that is, the standard heterogeneous agent NK model that does not feature the ZLB), and disentangle the overall change in that of taxes, interest earnings, labor earnings, and profit earnings. We then report these values across different levels of labor earnings – by considering the relative response of income across the three different realizations of the individual labor earnings process – and across different levels of wealth – by considering the relative response of individuals in the first decile of the wealth distribution with that of wealth-rich households, that is, households in the top 1% of the wealth distribution. We also report the aggregate relative response of income.

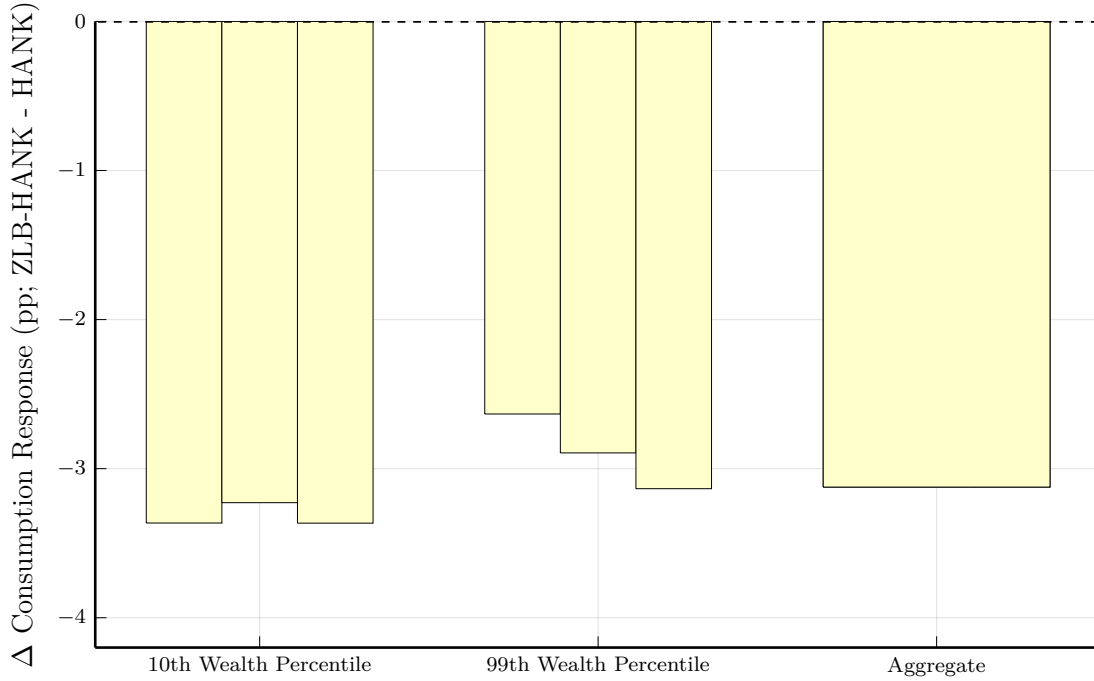
Figure 5: Individual Income Responses - ZLB-HANK vs. HANK.



Note: The graph reports the income response for each of the three labor-earning realizations at the 10th wealth percentile and the 99th wealth percentile, and then each bar decomposes the total income response in the changes in taxes, interest revenues, wages, and profits.

Figure 5 indicates several important results on the distributional effects of a recessionary shocks once accounting for the ZLB constraint. First, overall the response of income is negative, for any realization of labor earnings, any position in the wealth distribution, as well as when looking at the behavior of aggregate labor income. This is related to the results of the previous section, which suggest that for sufficiently large and negative realizations of

Figure 6: Consumption Responses - ZLB-HANK vs. HANK.



Note: The graph reports the consumption response for each of the three labor-earning realizations at the 10th wealth percentile and the 99th wealth percentile.

the demand shock, the drop in the nominal interest rate is so large to hit the ZLB constraint. In such a case, the monetary authority cannot reduce further its stance to ease the effects of the recession, and thus the drop in output, income, and consumption are exacerbated relative to the very same HANK economy that does not feature the ZLB constraint.

Second, the presence of the ZLB constraint amplifies the drop in the total income of wealth-poor households relative to that of wealth-rich individuals. Although both relative responses are negative, that associated to the households in the first decile of the wealth distribution is relatively larger. Again, this difference is due to the lack of the monetary authority of being able to stabilize the economy when the nominal interest rate hits zero. Indeed, we find what leads the drop in total income is the combination of the larger relative reduction in households' wages, and the relatively larger hike in interest payments. Thus, since the ZLB amplifies the drop in wages as a response to a recessionary demand shock – while raising the response of interest payments – it generates a differential impact along the wealth distribution such that the overall negative effects are concentrated on those individuals whose main income source are the labor earnings, that is, the wealth-poor households.

The presence of the ZLB alters the distributional consequences of business cycle fluctuations also when looking at the response of individual consumption. Figure 6 reports the

differences in the impact response of consumption in the ZLB-HANK and HANK economy to a negative demand shock, and it does so for the three level of the labor earnings process, as well as for the households in first decile of wealth distribution, the individuals at the top 1% of the wealth distribution, and the differential response of aggregate consumption. Once again, we find that the response of consumption is relatively more negative when accounting for the ZLB constraint, and once again this is amplified at lower level of the wealth distribution. Specifically, the presence of the ZLB amplifies the drop in consumption of wealth-poor individuals by about 3.5 percentage points.

Overall, this evidence suggests that accounting explicitly for the presence of the ZLB constraint alters the effects of business cycle fluctuations of a HANK economy both at the aggregate level – with recessions becoming much more severe upon large negative realizations of the demand shock process – and the individual level – with recessions whose burden becomes even larger for households at the lower end of the wealth distribution, thus increasing consumption, income, and wealth inequality during a severe downturn.

5.3 The Deterministic and Stochastic Steady States

Standard models tend to predict that the real interest rate in the deterministic steady state is pinned down by structural parameters, and so there is no role for monetary policy to influence it. What we will show next is that although our model features a similar result in the deterministic steady state, it generates a relationship between changes in the inflation target and changes in real interest rates in the stochastic steady state.

What is the difference between these two steady state concepts? The deterministic steady state is an equilibrium in which all variables are constant and households ignore any source of aggregate risk. Basically, in our case it represents the scenario in which the standard deviation of the demand shock goes to zero, that is, $\omega_\xi = 0$. Instead, the stochastic steady state (also known as the risky steady state, e.g. Coeurdacier, Rey and Winant, 2011)) describes the equilibrium in which all variables are constant and households take their optimal decisions by taking into account the aggregate risk and by explicitly considering that the standard deviation of the demand shock is positive, that is, $\omega_\xi > 0$, although no shock arrives along the equilibrium path.

The difference between these two steady states is then that in the deterministic steady state households do not anticipate the effect of future aggregate shocks, and this case is often referred to as the perfect foresight equilibrium. Instead, in the stochastic steady state households are well aware of the existence of future aggregate shocks that may hit the economy. In either case, households internalize the risk due to the idiosyncratic labor earning process.

Importantly, the stochastic steady state is a more accurate description of the long run behavior of non-linear models as ours. For instance, in the deterministic steady state of our model the level of the inflation rate is 2%, which coincides with the inflation target of the central bank. Instead, at the stochastic steady state the level of inflation is relatively lower, and equals 1.90%. This happens because households internalize the possible occurrence of large negative demand shocks that may bring the economy to the ZLB constraint, triggering deflationary episodes: the deflationary spiral channel. This examples shows the crucial relevance of considering the dynamics of the stochastic steady state of a model that features a non-linearity such as the ZLB, and in the next sections we will highlight how our model implies that in the stochastic steady state changes in the inflation target do affect the level of real interest rates.

We then leverage the differences in the value of key macroeconomic variables at the DSS and SSS across different model economies. Specifically, we compare first our ZLB-HANK model to an otherwise identical heterogeneous agent NK economy which does not feature the non-negativity constraint on the policy rate, that is, the standard HANK economy, and then we compare the ZLB-HANK model to an otherwise identical economy which abstracts from individual heterogeneity and thus features a representative households, the ZLB-RANK.

Table 2 reports the results of this comparison exercise for the ZLB-HANK, the HANK with no ZLB constraint, and the ZLB-RANK model, showing the DSS and SSS values for inflation, the nominal interest rate, the real interest rate, output, and the frequency – as well as the duration in quarters – of ZLB events. In the case of the HANK economy without the bound on policy rates, we report the shadow frequency and duration of ZLB events, defined as all those circumstances in which the nominal interest rate of the HANK economy equals to or is less than zero.

The table shows that the two economies are characterized by exactly the same values for all these key macroeconomic variables at the DSS. However, the values at the SSS differ across the two model versions. Specifically, the HANK economy that does not feature the ZLB constraint displays SSS values for the inflation, nominal rate, and real rate which are virtually equal to those at the DSS. Indeed, the difference amounts to 1 basis point (bp), 3 bps, and 2 bps, respectively. Instead, the introduction of the ZLB reduces the SSS values of inflation, the nominal rate, and the real rate by around 10 bps, 19 bps, and 9 bps, respectively.

What drives this difference between the two model economies at the SSS? The answer is the presence of the ZLB constraint, which alters the dynamics of inflation, the nominal and real rates. This can be observed also by comparing the frequency and duration of ZLB events across model economies: while the frequency and duration of ZLB occurrences equal 9.72%

Table 2: Comparison of DSS and SSS in ZLB-HANK, HANK, and ZLB-RANK.

Variable	ZLB-HANK		HANK		ZLB-RANK	
	DSS	SSS	DSS	SSS	DSS	SSS
Inflation	2.00%	1.90%	2.00%	1.99%	2.00%	1.91%
Nominal Rate	3.50%	3.31%	3.50%	3.47%	3.84%	3.67%
Real Rate	1.50%	1.41%	1.50%	1.48%	1.84%	1.75%
(Shadow) ZLB Frequency	-	9.72%	-	(5.88%)	-	8.36%
(Shadow) ZLB Duration in Quarters	-	1.61	-	(1.45)	-	1.66

and 1.61 quarters in our ZLB-HANK, these (shadow) moments are lower in the standard HANK economy, and equal to 5.88% and 1.45. As the ZLB does not allow the monetary authority to properly stabilize the economy when the nominal rate hits zero, this leads to more frequent and longer ZLB spells. Since the previous section shows that the ZLB affects relatively more the income and consumption of wealth-poor individuals, the ZLB-HANK economy is then characterized by a higher amount of households' precautionary savings, as individuals try to increase their buffer of savings (or equivalently, try to reduce their borrowing amounts) so to insure their consumption stream in those recessions in which the nominal rate hits zero. Higher precautionary savings exert a downward pressure on the level of the real interest rate, which then reduces the room of manoeuvre for the nominal rates of the central bank, and eventually makes the ex-post realization of the ZLB events even more likely.

When comparing the DSS and SSS values of the ZLB-HANK and the ZLB-RANK models, we find that although both economies share the same value for the inflation target in the DSS, the real interest rate in the DSS of the ZLB-HANK is 34 bps lower than that of the ZLB-RANK. In the RANK economy the real interest rate is uniquely pinned down by time discount factor β . The heterogeneous-agent counterpart features the very same parametrization – including that of the discount factor – but it is characterized by a lower DSS real rate because of households' precautionary savings against the idiosyncratic labor-

Table 3: Decomposition Exercise.

	Real Rate	Nominal Rate	Inflation
ZLB-RANK DSS	1.84%	3.84%	2.00%
ZLB-HANK SSS	1.41%	3.31%	1.90%
(i) Total	0.43pp	0.53pp	0.10pp
ZLB-RANK DSS	1.84%	3.84%	2.00%
ZLB-HANK DSS	1.50%	3.50%	2.00%
(ii) Precautionary Savings Idiosyncratic Risk	0.34pp	0.34pp	0.0pp
(i)-(ii) Deflationary Bias	0.09pp	0.19pp	0.10pp

earning risk.⁵ Although this result traces back to the seminal work of Aiyagari (1994), our setting grants it a novel perspective, as it generates another dimension through which households' precautionary savings reduce the room of manoeuvre for the central bank's policy rate. In this way, the precautionary savings in the DSS alter the ZLB frequency, and consequently affect also the behavior of the macroeconomic variables in the SSS. Indeed, the ZLB-HANK economy has a frequency of ZLB events which is around 10%, above the 8.36% frequency generated by the RANK counterpart. Importantly, this result does not emerge in the standard HANK literature, in which the drop in the real rate due to precautionary savings is immaterial for aggregate dynamics because of the lack of the ZLB.

⁵At the DSS households do not require to increase their buffer of savings against the realization of aggregate risk, as this equilibrium ignore any realization of the aggregate demand shock. However, households still face the additional idiosyncratic uncertainty due to the uninsurable labor earning earning process.

Overall, the analysis of the DSS and the SSS shows that the real rate in the ZLB-HANK economy is endogenously determined by the interaction of the presence of the ZLB constraint and households' heterogeneity. However, the overall effect compounds two different channels that curtail real rates: the *precautionary savings against their idiosyncratic labor-earning risk* and the *deflationary bias* channel highlighted by Adam and Billi (2007), Nakov (2008), Hills, Nakata and Schmidt (2019), and Bianchi, Melosi and Rottner (2020). In the first case, households raise their savings buffer to cope with the low realizations of labor earnings. Instead, the second channel works through the risk of hitting the ZLB: when the inflation target is sufficiently low and the probability of hitting the ZLB is relatively high, agents correctly expect that the ZLB may be binding in the future and internalize that the central bank may not be able to fully stabilize inflation. As a result, nominal interest rates decrease, reducing real rates, and eventually increasing the likelihood of hitting the ZLB even more.

To disentangle the two channels, we propose in Table 3 a simple decomposition exercise. To measure the relevance of precautionary savings due to idiosyncratic risk, we compare the DSS values of the ZLB-RANK economy with those of our baseline heterogenous-agent model. This channel leads to a 34 bps drop in the real rate, whereas it does not affect the inflation rate, as inflation target in both the HANK and RANK economies equals 2%. As a result, we also find a 34 bps bias for the nominal interest rate.

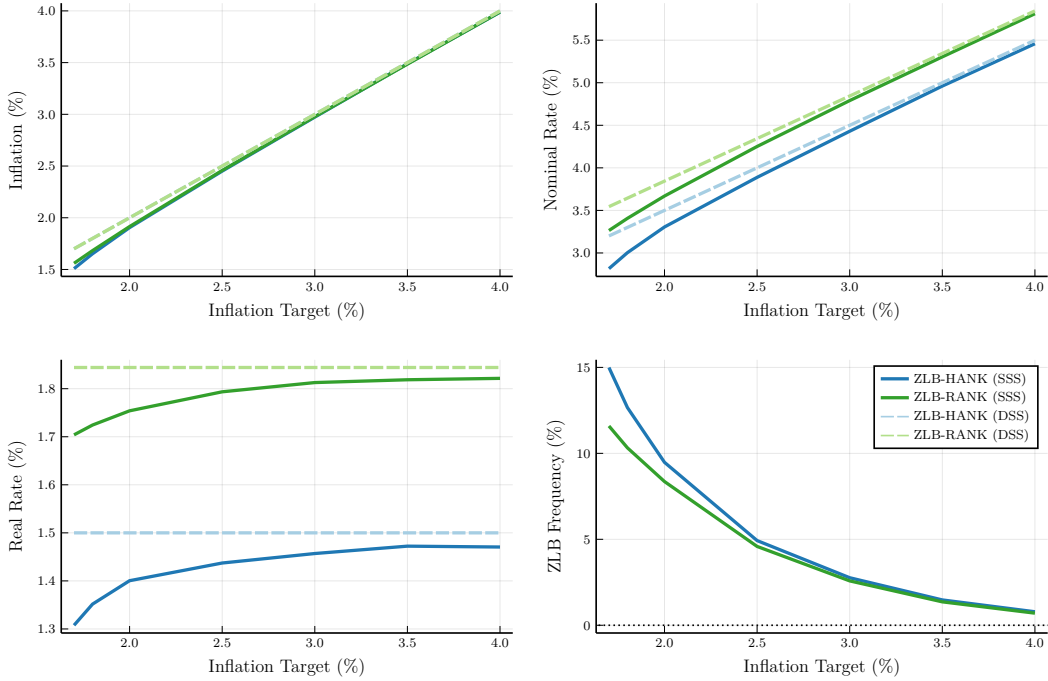
The role of the deflationary bias can be isolated by subtracting the contribution of the precautionary savings due to idiosyncratic risk from the total change in real rates between the SSS of the ZLB-HANK economy and the DSS of the ZLB-RANK economy. We find that the deflationary bias reduces the level of real rates, nominal rates, and inflation by 9 bps, 19 bps, and 10 bps, respectively.

In the next two sections, we show that the downward bias in the level of the real interest rate in our ZLB-HANK economy depends crucially on the interaction between the inflation target set by the monetary authority and the amount of wealth heterogeneity, such that the bias gets substantially amplified in economies with relatively lower inflation targets and relatively larger inequality.

5.4 Inflation Target and Real Interest Rates

Instead, in our environment the precautionary savings amplify the deflationary spiral channel, through which the equilibrium level for the real rate depends on households' expectations about the costs of experiencing a ZLB event. Since the monetary authority can alter the frequency of the ZLB spells by modifying its inflation target, it ends up affecting households' expectations and eventually the level of real interest rates. In other words, monetary policy is not neutral. More specifically, the model features a long-run Fisher equation,

Figure 7: DSS/SSS in ZLB-RANK and ZLB-HANK as a Function of the Inflation Target.



$i(\tilde{\pi}) = r(\tilde{\pi}) + \pi(\tilde{\pi})$, in which the real rate in the stochastic steady state depends on the central banks' inflation target $\tilde{\pi}$. In this setting, a higher inflation target raises the SSS level of the real rate, that is, $dr/d\tilde{\pi} > 0$.

To establish this result, we compare the level of the real interest rate in different model economies, which uniquely differ in the parameter $\bar{\pi}$, that is, the level of the inflation target of the monetary authority. Figure 7 reports the DSS and SSS levels of the real rate, the nominal rate, and the frequency of the ZLB for inflation targets between 1.7% and 4% in both the ZLB-HANK economy, as well as the ZLB-RANK model. First, the figure shows that – in both the RANK and HANK economy – when the inflation target is between 3% and 4%, then the central bank manages to achieve its inflation target in the stochastic steady state. Indeed, the inflation rate in the stochastic steady state almost coincides with that at the deterministic steady state. Moreover, the probability of experiencing a ZLB event is so low that the non-linearity in the setting of the nominal interest rate is not a quantitatively relevant force in shaping households' expectations and optimal decisions. Accordingly, the economy behaves as if there is no ZLB constraint, and changes in the inflation target do not alter the level of real interest rates.

However, when the inflation target of the central bank goes below 3%, then the non-linearity due to the ZLB kicks in, and the implications of the stochastic steady state diverge substantially from those of the deterministic steady state. First of all, the central bank start

undershooting its inflation target: when the target is 1.7%, the inflation rate in the stochastic steady state equals 1.51%. This 0.19 percentage points undershooting comes together with a very high likelihood that the economy experiences a ZLB event. Indeed, the probability of hitting the floor in the nominal interest rate raises from below 1% when the inflation target is 4% up to 15% when the inflation target is 1.7%. The ZLB frequency is relatively less sensitive to changes in the inflation target in the RANK model, as it raises from below 1% when the inflation target is 4% up to 11.6% when the inflation target is 1.7%.

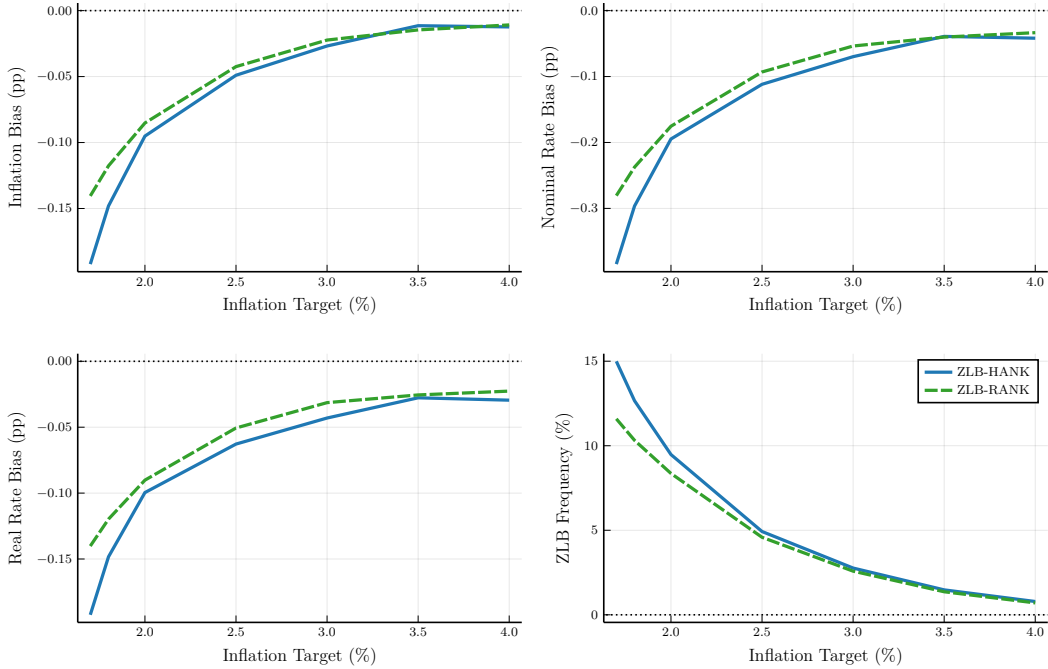
In this exercise, we find that although the DSS level of the real interest rate is lower in the HANK economy than in the RANK – due to the precautionary savings against the idiosyncratic labor earning risk – in each economy the DSS level of the real rate is constant and independent of the monetary policy stance. Instead, at the SSS there is a relevant quantitative relationship between the inflation target and real interest rates when the target is below 3%. Indeed, we find that although the level of real rates is very close to the 1.5% of the deterministic steady state when the target is above 3%, the real rates drop substantially as the target shrinks. When the inflation target reaches 1.7%, then the level of real rates fall to 1.31%, amounting to roughly a 16 basis point drop from the level of real rates associated with the 4% inflation target. This is a novel prediction of the model which crucially hinges on incorporating explicitly the non-linearity of the ZLB constraint in the dynamics at the stochastic steady state. If we were to abstract from the ZLB or consider the deterministic steady state dynamics, then the model would predict the standard result that the monetary policy stance does not affect the level of the real interest rate.

The relationship between the level of the real rate at the SSS and the level of the inflation target is slightly less powerful in the RANK model, that is, the magnitude of the derivative $dr/d\tilde{\pi}$ in RANK is lower than the same derivative $dr/d\tilde{\pi}$ in HANK. All these dynamics can be better appreciated in Figure 8, which reports the differences between the SSS and DSS values of the RANK and HANK economies at each value of the inflation target. Specifically, the drop in the real rate in the ZLB-RANK going from an inflation target of 4% to 1.7% is 4 bps lower than in the ZLB-HANK economy.

What drives the relationship between the inflation target and the real interest rates? To inspect this relationship, let us refer again to the Fisher equation. In the steady state, standard models pin down the level of the real rate as the inverse of the households' time discount parameter, whereas the level of inflation is a policy parameter. Thus, given these two structural parameters, standard models recover the level of the nominal interest rate.

Although our economy features the same property in its deterministic steady state, the dynamics change in the stochastic steady state. In this case, the values of both the inflation rate and the real interest rate are endogenously determined, so that the model features an

Figure 8: Differences between the SSS and DSS as a Function of the Inflation Target.



equilibrium long-run Fisher equation. Namely, the link between the inflation target and the real interest rate is twofold. On the one hand, the relationship works through changes in households' precautionary savings. As the probability of ZLB events is almost as high as 15% with a 1.7% inflation target, and those events affect disproportionately more wealth-poor agents, a decline in the inflation target which raises the probability that policy rates become constrained has the implication of raising households' precautionary savings. As a result, the interest rate declines. We will provide further evidence on this mechanism in the next section.

Importantly, the non-neutrality of the inflation target arises already in the RANK model, as indicated in Figures 7 and 8. In this economy, the inflation target alters real rates through a deflationary spiral channel already highlighted by Adam and Billi (2007), Nakov (2008), Hills, Nakata and Schmidt (2019), and Bianchi, Melosi and Rottner (2020): when the inflation target is sufficiently low and the probability of hitting the ZLB is relatively high, agents correctly expect that the ZLB may be binding in the future and internalize that the central bank may not be able to fully stabilize inflation. As a result, nominal interest rates decrease, reducing real rates, and eventually increasing the likelihood of hitting the ZLB even more. Thus, this mechanism self reinforces itself, with the ultimate implication that reductions in the inflation target may substantially raise the probability of hitting the ZLB and reduce the level of real rates. While the non-neutrality of the inflation target in the

RANK economy is quite limited, in the next section we show that changes in the inflation target can lead to quantitatively relevant variations in the real rate at high level of wealth inequality.

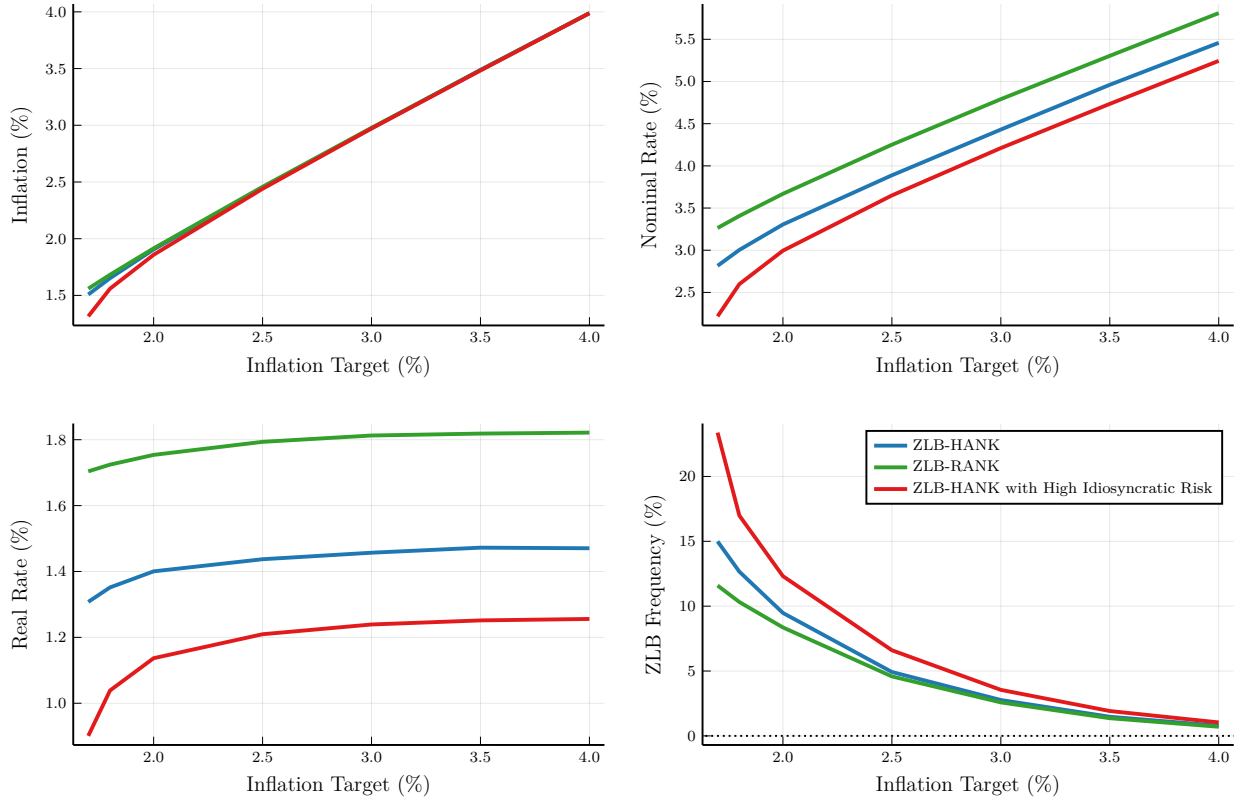
5.5 The Role of Wealth Inequality

What is the role of household heterogeneity in generating the relationship between changes in the inflation target and the variation in the real interest rate at the stochastic steady state? The non-neutrality of monetary policy crucially depends on the interaction of the inflation target with the amount of wealth inequality in the economy. To establish this result, we compare the effects of changes in the inflation target of our baseline ZLB-HANK economy and its ZLB-RANK counterpart with one additional ZLB-HANK economy which differs from the benchmark case for the amount of wealth inequality. More specifically, we increase the standard deviation of the labor earning risk process, ω_s , such that the Gini index goes from the value of 0.81 our baseline economy to 0.84. This increase in the Gini index of wealth corresponds to the change observed in the U.S. economy in the early 2000s (Kuhn and Ríos-Rull, 2016). Figure 9 reports the SSS values for the inflation target, the nominal interest rate, the real rate, and the ZLB frequency for the baseline HANK model, the RANK counterpart, and the additional economy with higher idiosyncratic risk.

We find that the non-neutrality of monetary policy does depend substantially on the level of wealth inequality: when we increase the amount of wealth inequality with the higher level of idiosyncratic risk, then the drop in the real rate between the 4% and 1.7% inflation targets amounts to 36 bps, a change which almost halves the level of the real rate. Thus, our mechanism can generate a variation in the real rate which is three times as large as that of the deflationary spiral channel, and thus suggest that accounting for households' inequality is key in generating a quantitatively relevant long-run Fisher equation, in which the monetary policy stance does affect the long-run values of the key macroeconomic aggregates.

Why does a higher wealth inequality lead to a larger drop in the real rate for lower levels of the inflation target? Again, the answer lies in the role of the precautionary savings. Households accumulate an additional buffer of savings (or alternatively, reduce their borrowing exposure) to shield their consumption stream during a ZLB recession. This is especially important for low-wealth individuals, who bear the burden of these severe downturns. In the low idiosyncratic risk economy, the relatively low incidence of the ZLB implies that households do not boost their precautionary savings at lower levels of the inflation target. However, in the high idiosyncratic risk economy the frequency of the ZLB gets as high as 23% at the 1.7% inflation target. Since the ZLB is visited once every four quarters on average and the higher level of wealth inequality means that there are many more individuals

Figure 9: The Interaction Between the Inflation Target and Wealth Inequality.



at the lower end of the distribution who bear the costs of the ZLB spells, households decide to substantially increase their saving buffers, leading to a drop in the level of the real rate.

Overall, these results are consistent with the observed dynamics of the U.S. economy over the recent decades. As the level of the average inflation rate has shrunk from around 4% in the 1980s and 1990s to below 2% in the 2000s – and these changes have happened contemporaneously with a secular rise in the amount of wealth inequality – then the economy has started experiencing ZLB events, the inflation rate has been constantly below the 2% target, and the level of real interest rates has dropped substantially. Thus, our results provide a novel rationale that could comprehensively explain all these events jointly.

6 Concluding Remarks

This paper shows how to explicitly incorporate the presence of the ZLB constraint into a heterogeneous agent New Keynesian economy. To do so, we propose a novel neural network algorithm that allows to solve for the fully non-linear stochastic equilibrium. In this way, we can show that the presence of the ZLB constraint substantially alters the dynamics and the properties of the HANK economy.

First, the presence of the ZLB alters the response of aggregate output, inflation, and the real rate to large shocks. In the cases that the nominal rate hits the ZLB, the monetary authority loses its stabilization power. As a result, the drop in output is twice as large as in a model without the constraint on the policy rates, and it generates a relatively more muted response on the real interest rate, accompanied by a larger decrease in the wage. Thus, the presence of the ZLB further depresses labor income vis-à-vis capital income during a severe recession.

Second, the presence of the ZLB amplifies the negative recessionary effects on income and consumption for wealth-poor households, as the income of these individuals is disproportionately tilted towards labor earnings. In addition, wealth-poor households do not have a sufficiently large buffer to insure their consumption stream during a downturn. Consequently, the ZLB leads to a larger increase in income and consumption inequality during severe recessions.

Third, households react to the additional risk posed by the presence of the ZLB by raising their precautionary savings, even above the level that characterizes any heterogeneous agent economy at its deterministic steady state. Overall, the precautionary savings exert a downward pressure on interest rates, thus reducing the room of manoeuvre for the central banks policy stance. This behavior leads to an even larger ex-post equilibrium frequency and duration of the ZLB events.

Fourth, we uncover a novel relationship between the stance of monetary policy – and namely, the inflation target set by the central bank in its Taylor rule – and the level of real interest rates. Usually, the level of real interest rates in standard New Keynesian models is pinned down by structural parameters, and so although the level of real rates may shape the optimal conduct of monetary policy, there is no feedback effect coming out from between monetary policy decisions. Instead, in our environment the precautionary savings lead to an equilibrium level for the real rate which depends on households' expectations about the costs of experiencing a ZLB event. Since the monetary authority can alter the frequency of the ZLB spells by modifying its inflation target, it ends up affecting households' precautionary savings and eventually the level of real interest rates. In other words, monetary policy is not neutral. This would not be the case if we were to derive the model in a perfect foresight equilibrium, in which the non-linearity of the ZLB are plugged in the model but without properly affecting households' expectations and optimal decisions.

Finally, the non-neutrality of monetary policy is largely amplified at higher values of wealth inequality. The model predicts that the interaction between a drop in the inflation target from from 4% to 1.7% – which corresponds to the variation in the average inflation rate over the last three decades – and an increase in the Gini index of wealth of three percentage

points – which corresponds to the observed rise in wealth inequality in the early 2000s – reduces the level of the real interest rate by around 35 basis points. This decline in the level of the real interest rate explains around 20% of the 200 basis point reduction over the last decades that has been recently estimated by Del Negro et al. (2017) and Fiorentini et al. (2018). Hence, we provide a novel explanation for the drop in real interest rates, such that this decline may not be only due to demographic factors, reduction in the productivity of the economy, or variation in the convenience yields demanded by investors, but also may be a byproduct of the more aggressive stance of the central bank in reducing the level of the inflation target.

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A Computational Algorithm

Our novel computational algorithm for solving a fully nonlinear HANK economy is based on the stochastic simulation algorithm in Maliar, Maliar and Valli (2010). In contrast, the bond distribution is represented as a histogram following the procedure of Young (2010). Dropping indexes when not necessary, we start with the Euler equation of the individual household problem:

$$\left(c - \chi \frac{h^{1+\nu}}{1+\nu}\right)^{-\sigma} - \mu = \beta \mathbb{E} \left(\frac{\xi'}{\xi} \frac{R}{\pi'} \left(c' - \chi \frac{h'^{1+\nu}}{1+\nu}\right)^{-\sigma} \right) \quad (\text{A.1})$$

where μ is the Lagrange multiplier associated with the borrowing constraint of the households. Using the budget constraint, we can rewrite this condition as

$$\begin{aligned} \tilde{b}' &= \frac{R_{-1}}{\pi} b + wsh + \Pi s \\ &- \left[\mu + \beta \mathbb{E} \left(\frac{\frac{\xi' R}{\xi \pi'}}{\left(\frac{R}{\pi'} b' + w' s' h' - \tau (w' s' h')^{1-\gamma} + \Pi' s' - b'(b') - \chi \frac{h'^{1+\nu}}{1+\nu}\right)^\sigma} \right) \right]^{-1/\sigma} \\ &- \chi \frac{h^{1+\nu}}{1+\nu} - T, \end{aligned} \quad (\text{A.2})$$

where $\mu \equiv \mu(b, R_{-1}, s, \xi)$, $b' \equiv b'(b, R_{-1}, s, \xi)$, and $b'(b') \equiv b'(b'(b, R_{-1}, s, \xi))$. We construct grids of points for $b \in [b_{min}, b_{max}]$ and $R \in [R_{min}, R_{max}]$, in addition to the grids for the idiosyncratic state $s \in \{s_m\}_{m=1}^M$ and the aggregate state $\xi \in \{\xi_j\}_{j=1}^J$.⁶ The individual problem is then solved on this grid according to the following algorithm.

Algorithm 1 (Individual Problem)

1. Make a guess for the bond policy function $b'(b, R_{-1}, s, \xi)$ on the grid. We set the initial bond policy function to the DSS bond policy function for all aggregate states (R_{-1}, ξ) .
2. For each grid point (b, R_{-1}, s, ξ) , substitute the assumed bond policy function $b'(b, R_{-1}, s, \xi)$ in the right hand side of (A.2), set the Lagrange multiplier equal to zero, and compute the new bond policy function in the left-hand side of equation (A.2). The required labor policy is $h = \left(\frac{\tau(1-\gamma)}{\chi} (ws)^{1-\gamma}\right)^{\frac{1}{\nu+\gamma}}$. For each point that does not belong to $[b_{min}, b_{max}]$, set $\tilde{b}'(b, R_{-1}, s, \xi)$ equal to the corresponding boundary value.

⁶We use the polynomial rule from Maliar, Maliar and Valli (2010) for b to place more grid points near the borrowing constraint. The grid for R is evenly spaced, however. In economies with a ZLB, we can set $R_{min} = 1$.

3. Compute the bond function for the next iteration $\tilde{\tilde{b}}(b, R_{-1}, s, \xi)$ using:

$$\tilde{\tilde{b}}(b, R_{-1}, s, \xi) = \lambda_b \tilde{b}'(b, R_{-1}, s, \xi) + (1 - \lambda_b) b'(b, R_{-1}, s, \xi) \quad (\text{A.3})$$

where $\lambda_b \in (0, 1]$ is an updating parameter.

Iterate on steps 2 and 3 until the maximum difference between $\tilde{\tilde{b}}(b, R_{-1}, s, \xi)$ and $\tilde{b}'(b, R_{-1}, s, \xi)$ is less than a given degree of precision.

As discussed in Maliar, Maliar and Valli (2010), the algorithm satisfies the Euler equation, the budget constraint, and the complementary slackness condition by construction.

When solving the individual problem, one also needs several “aggregate” variables, such as π_t , τ_t , w_t , and Π_t . We use Krusell and Smith (1998) and Fernández-Villaverde, Hurtado and Nuño (2020) to build PLMs that predict $\log \pi_t$ and $\log \left(\frac{\pi_{t+1}}{\pi} \right) \frac{Y_{t+1}}{Y_t}$, given the aggregate states (R_{t-1}, ξ_t) using neural networks and moments of the household distribution. Given this, we can back out m_t from the Philips curve and all other “aggregate” variables required for solving the individual problem.⁷

The initial guesses for our PLMs are such that initially inflation is at its target and the inflation expectation term is such that marginal costs are always at their DSS value, i.e., $m_t = \frac{\varepsilon-1}{\varepsilon}$. Given this, the whole model can be solved as follows.

Algorithm 2 (Stochastic Simulation)

1. Generate and fix a time series of length T for the aggregate shocks.
2. Set initial matrices D_i ($i \in \{\pi, \mathbb{E}\pi\}$) for the PLMs and initial distribution of bonds. We use the distribution of bonds in the DSS and represent this distribution as a histogram as in Young (2010).⁸
3. Given the PLMs as represented by D_i , compute a solution to the individual problem as described in Algorithm 1. Off-grid values of D_i are linearly interpolated.

⁷For comparison, we also found PLMs by computing linear regressions:

$$\begin{aligned} \log \pi_t &= \beta_{(0,\pi)} + \beta_{(1,\pi)} \log R_{t-1} + \beta_{(2,\pi)} \log \xi_t + \beta_{(3,\pi)} \log R_{t-1} \log \xi_t \\ \log \left(\frac{\pi_{t+1}}{\pi} \right) \frac{Y_{t+1}}{Y_t} &= \beta_{(0,E\pi)} + \beta_{(1,E\pi)} \log R_{t-1} + \beta_{(2,E\pi)} \log \xi_t + \beta_{(3,E\pi)} \log \xi_{t+1} \\ &\quad + \beta_{(4,E\pi)} \log R_{t-1} \log \xi_t + \beta_{(5,E\pi)} \log R_{t-1} \log \xi_{t+1} + \beta_{(6,E\pi)} \log \xi_t \log \xi_{t+1}. \end{aligned}$$

These are similar to the specification in Bayer et al. (2019) but adapted to the fact that the aggregate state ξ_t follows an AR(1) process and not a Markov chain.

⁸ D_π and $D_{E\pi}$ are matrices that represent the predictions of the neural network (evaluated on dense grids for R_{t-1} , ξ_t , and ξ_{t+1}). We do not use the prediction of the neural network directly to be able to update the PLMs slowly.

4. Use the individual policy functions, the PLMs, and the aggregate shocks from step 1. to simulate the economy forward as in Young (2010) and calculate average bond holdings B_t . If $B_t \neq \tilde{B}$, use a nonlinear solver to find π_t that implies $B_t = \tilde{B}$. This requires updating individual policies where we take the PLM for inflation expectations and all $t + 1$ policies as given.
5. Train the neural networks on the simulated data (use the series of π_t and R_{t-1} that imply $B_t = \tilde{B}$ at each simulation step).
6. Evaluate the neural network on a dense grid of points. Let \tilde{D}_i be the resulting matrices.
7. Compute the PLMs for the next iteration using the updating formula:

$$\tilde{\tilde{D}}_i = \lambda_{PLM} \tilde{D}_i + (1 - \lambda_{PLM}) D_i \quad (\text{A.4})$$

where $\lambda_{PLM} \in (0, 1]$ is an updating parameter.

Iterate on steps 3.-7. until the average squared difference between $\tilde{\tilde{D}}_i$ and D_i is less than a given degree of precision.