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Two-State Quantum Asymptotics

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Only wimps specialize in the general case. Real scientists pursue examples.
—Adapted from a remark by Beresford Parlett

INTRODUCTION

John A. Wheeler's distinctive intellectual stamp includes the choice of simple examples to demystify difficult and subtle ideas, and I will follow that lead here. In the last decade, we have learned about several general quantum phenomena that share the property that they are emergent as a parameter (ϵ in what follows) vanishes. Remarkably, all can be illustrated by the simplest nonsimple quantum problem, namely, the evolution of two-state systems with a time-dependent Hamiltonian.

Such systems are governed by the Schrödinger equation,

$$i\partial_t|\psi\rangle = \mathbf{R}(\epsilon t)\cdot\mathbf{S}|\psi\rangle. \tag{1}$$

Here, the state is the 2-spinor

$$|\psi\rangle = |\psi(t,\epsilon)\rangle = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}; \tag{2}$$

S is the vector of spin-½ matrices,

$$\mathbf{S} \equiv \left(\frac{1}{2}\right)\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]; \tag{3}$$

and the vector that drives the system is

$$\mathbf{R}(\epsilon t) \equiv \{X(\epsilon t), Y(\epsilon t), Z(\epsilon t)\}. \tag{4}$$

$\mathbf{R}(\epsilon t)$ will be called the Hamiltonian vector and its track over $-\infty < t < +\infty$ will be called the Hamiltonian curve. The Hamiltonian itself is

$$\mathbf{H}(\epsilon t) = \mathbf{R}(\epsilon t)\cdot\mathbf{S} = \left(\frac{1}{2}\right)\begin{pmatrix}Z(\epsilon t) & X(\epsilon t) - iY(\epsilon t)\\ X(\epsilon t) + iY(\epsilon t) & -Z(\epsilon t)\end{pmatrix}. \tag{5}$$

Vanishing asymptotic parameter $\bullet$ is the adiabatic limit of slow driving (except for the final example in the last section below, where ϵ will have a different interpretation). This two-state formalism has several physical interpretations; one is the spin state of a neutron in a slowly changing magnetic field $\mathbf{R}(\epsilon t)$.

In each of the five examples to follow, I shall first give a brief description of the general phenomenon and then proceed to its two-state illustration.

GEOMETRIC PHASE

A quantum system driven by parameters that are slowly changed round a cycle C will, because of the quantum adiabatic theorem,[1] cling closely to an eigenstate of the instantaneous Hamiltonian and thus return to its original state, apart from a phase factor. This phase factor[2–6] is

$$\langle \psi_n(\text{beginning}) \,|\, \psi_n(\text{end}) \rangle = \exp\left\{ \left(-\frac{i}{\epsilon} \right) \int_{\text{cycle}} d\tau E_n(\tau) \right\} \exp\{ i\gamma_n(\text{C}) \}. \tag{6}$$

Here, n labels the state being transported round C. The first factor, involving the instantaneous energy $E_n(\epsilon t)$, contains the dynamical phase, generalizing the $-i\omega t$ in the evolution of any wave. The second factor contains the geometrical phase, given by the line and surface integrals

$$\gamma_n(\text{C}) = -\text{Im} \oint_{\text{C}} \langle n \,|\, dn \rangle = -\text{Im} \int\int_{\partial S = \text{C}} \langle dn \,|\wedge|\, dn \rangle, \tag{7}$$

where $|n\rangle$ is the instantaneous eigenstate, with the phase chosen so that $|n\rangle$ is single-valued round C.

For a two-state system, the parameters are the components (equation 4) of the Hamiltonian vector, C is a loop in **R** space, and n labels the state with energy $(\pm\tfrac{1}{2})|\mathbf{R}|$ [that is, spin $(\pm\tfrac{1}{2})\hbar$]. Then, the geometric phase is[2]

$$\gamma_\pm(\text{C}) = (\mp\tfrac{1}{2})\Omega(\text{C}), \tag{8}$$

where Ω is the solid angle subtended by C at $\mathbf{R} = 0$. This version of the geometric phase has been measured with a neutron beam,[7] along which the direction of a magnetic field is varied round a cone. A different implementation was known much earlier[8–11] for light beams whose state of polarization was cycled; in this case, parameter space is the Poincaré sphere of polarizations, that is, $|\mathbf{R}| = 1$.

The geometric phase has been generalized to nonadiabatic cycle evolutions,[12] for which the state returns exactly apart from a phase factor (this can be made to happen even if the Hamiltonian does not depend on time). Then, equation 8 still holds, but now Ω is the solid angle of the loop traversed by the vector $\langle \psi(t) \,|\, \mathbf{S} \,|\, \psi(t) \rangle$ (which coincides with **R** only adiabatically).

However, the adiabatic case contains surprising richness, in the form of corrections[13] to equation 6 containing higher powers of ϵ. When ϵ is small, but not zero, the state does not return exactly. Instead, we have

$$\langle \psi_n(\text{beginning}) \,|\, \psi_n(\text{end}) \rangle = a(\epsilon) \exp\{ i\phi(\epsilon) \}. \tag{9}$$

If the Hamiltonian curve is analytic, the deviation from unity of the modulus $a(\epsilon)$, corresponding to transitions out of the initial state, is exponentially small, that is, beyond all orders, in ϵ. However, the phase $\phi(\epsilon)$ does have a power series in ϵ, whose first two terms (those not vanishing in the adiabatic limit) are given in equation 6. The geometric phase is the term not involving ϵ and can be expressed as an adiabatically emergent phenomenon according to

$$\gamma_n(C) = \lim_{\epsilon\to 0}\left(\frac{d}{d\epsilon}\right)[\epsilon\phi(\epsilon)]. \tag{10}$$

An iteration method[13] for calculating higher orders of the expansion consists of successive ("superadiabatic") transformations to moving frames, aimed at freezing the Hamiltonian vector $\mathbf{R}(t)$. These are time-dependent transformations, which change the Hamiltonian, so the freezing is not quite successful. Instead, C transforms to a sequence of renormalized loops that initially get rapidly smaller and whose geometric phases are the corrections to $\gamma_n(C)$; the r-th such correction is of order ϵ^r. The sequence diverges and the divergence has a universal form[13] related to the exponentially small transition probability.

GEOMETRIC AMPLITUDE

If the adiabatically varied Hamiltonian parameters depend analytically on time—and now the Hamiltonian curve need not be closed—the exponentially small final probability for transitions out of the initial state also contains, in general, a geometric correction,[14,15] independent of ϵ. This was as unexpected as the geometric phase, of which it is an analytic continuation. The geometric amplitude is associated with complex times for which the instantaneous eigenstates are degenerate (it being assumed that there are no real degeneracies); these degeneracies are the source of the weak adiabatic transitions.

For two-state systems, degeneracies arise from complex times τ^* when

$$R(\tau^*) \equiv [X^2(\tau^*) + Y^2(\tau^*) + Z^2(\tau^*)]^{1/2} = 0. \tag{11}$$

The transition probability from (say) the initial state $|\psi(-\infty)\rangle = |+(-\infty)\rangle$, with energy $+R$, to that with $-R$ is[14]

$$|\langle -(+\infty)|\psi(+\infty)\rangle|^2$$
$$= \exp\left\{\left(-\frac{2}{\epsilon}\right)\left|\operatorname{Im}\int_0^{\tau^*} d\tau R(\tau)\right|\right\}\exp\left\{-2\operatorname{Im}\int_0^{\tau^*} d\tau\dot{\phi}(\tau)\cos\theta(\tau)\right\}. \tag{12}$$

Here, $\theta(\tau)$ and $\phi(\tau)$ are the polar angles of the Hamiltonian curve. The first ("dynamical") exponent was familiar[16,17] and is the dominant contribution to the exponentially weak transitions. The second exponent is geometric and can be regarded as a complex generalization of the solid angle in equation 8.

It follows from equation 12 that the geometric exponent is zero if the Hamiltonian curve lies in a plane through $\mathbf{R} = 0$ (as in the familiar Landau-Zener problem[16]) or is reversible in the sense that it can be rigidly rotated about an axis through $\mathbf{R} = 0$ so as to coincide with its time-reverse (this case includes uniform helices). The simplest curve whose geometric exponent is not zero is the "winding-unwinding" helix, for which

$$\mathbf{R}(\tau) = \{\Delta\cos B\tau^2, \Delta\sin B\tau^2, A\tau\}. \tag{13}$$

In this case, the geometric amplitude factor is[14]

$$\exp\left\{-\pi\left(\frac{B\Delta^2}{A^2}\right)\operatorname{sgn}(A)\right\}. \tag{14}$$

The successful observation of the geometric amplitude in a magnetic resonance experiment[18] confirms that it describes real physics in the complex plane and is an asymptotically emergent phenomenon, separating from the dynamical contribution as $\epsilon \to 0$.

WKB PHASES, BOSONS, AND SEMIONS

In the early years of the geometric phase, it was asked whether the phase π responsible for the "½" of the "n + ½" WKB (semiclassical) quantization of oscillators (exact in the harmonic case) is geometric. My first answer was no, but this was wrong. It turns out that this semiclassical π [and related phases in optics, such as Gouy's (1899)[19] $\pi/2$ jump through a focus and Stokes' (1847)[20] similar jump through a caustic] does correspond to a geometric phase.

One way to see this is to cast the one-dimensional stationary Schrödinger equation for a nonrelativistic particle in a potential well $V(z)$, namely,

$$\left(\frac{\hbar^2}{2m}\right)\left(\frac{d^2}{dz^2}\right)\psi(z) + [E - V(z)]\psi(z) = 0, \tag{15}$$

in the form of a two-state problem. Defining

$$\epsilon \equiv \frac{\hbar}{\sqrt{2m}}, \qquad z \equiv \epsilon t, \qquad \psi_1 \equiv \psi, \qquad \psi_2 \equiv \frac{d}{dt}\psi, \tag{16}$$

we find equations 1–4 with

$$X - iY = 2i, \qquad X + iY = 2i[V(\epsilon t) - E], \qquad Z = 0 \tag{17}$$

and with the adiabatic limit now being interpreted as the semiclassical limit $\hbar \to 0$.

The semiclassical quantum condition, determining the allowed energies E, is that the adiabatic eigenstates specified by equations 16 and 17, which correspond to left- and right-moving waves, should be single-valued after a circuit of the two classical turning points defined by $V(z) = E$. From equation 17, these are the degeneracies of the matrix **H** in equation 5. The quantum condition includes two contributions: a dynamical phase, given by the classical action of the circuit divided by $\hbar$, and a geometric phase associated with the degeneracies. An immediate difficulty is that **H** is not Hermitian, but this is resolved by a generalization[21] of the first half of equation 7, namely,

$$\gamma_n(\mathrm{C}) = i\oint_{\mathrm{C}} \frac{\langle\tilde{n}|dn\rangle}{\langle\tilde{n}|n\rangle}, \tag{18}$$

where $\langle\tilde{n}|$ is the left eigenvector of equation 5 (that is, the row vector given by the conjugate transpose of the eigenvector of $\mathbf{H}\dagger$).

Application of equation 17 shows[22] that equation 18 has two interesting properties. First, it is real and thus does represent a phase. This is surprising because, for general non-Hermitian matrices, equation 18 will give a complex number. Second, it is the same for both of the degenerating states, rather than opposite as in equation 8. This is a consequence of the fundamental property that the eigenvectors of a non-Hermitian matrix become parallel at a degeneracy, rather than remaining orthogonal as in the Hermitian case (the same property has interesting consequences elsewhere in physics—for example, in the optics of absorbing crystals[11]). A calculation shows that the phase is

$$\gamma(C) = \left(-\frac{\pi}{2}\right) \times [\text{number of zeros of } E - V(z) \text{ inside C}]. \tag{19}$$

For an oscillator, there are two turning points, showing that the WKB π phase can indeed be interpreted as geometric.

In the Hermitian case of the second section, a planar circuit of a degeneracy gives the familiar π geometric phase associated with rotation of a spin-½ particle. In the present non-Hermitian case, the $-\pi/2$ phase associated with each degeneracy could also be achieved by planar rotation of a spin-¼ particle, that is, a semion. Given the connection of harmonic oscillators with bosons, we arrive at the curious suggestion that a boson can, in a sense, be regarded as made of two semions.

STOKES' PHENOMENON AND ADIABATIC QUANTUM JUMPS

Asymptotics, that is, the study of divergent series associated with singular limits, is currently enjoying worldwide resurgence. The deepest reason for this is the recognition[23] that relations between physical theories (e.g., quantum and classical mechanics, and statistical mechanics and thermodynamics) take the form of limits (e.g., $\hbar \to 0$ or $N \to \infty$) and these limits are usually singular. A central feature of divergent series is that their divergent tails often yield information about exponentially small terms, which are beyond all orders[24] of the series and whose existence is the cause of the divergence. As variables (not the asymptotic parameter) are changed, the small exponentials can appear and disappear; this is Stokes' phenomenon.[25]

Recent progress in asymptotics[26,27] has enabled Stokes' phenomenon to be understood in detail, with results that will now be described (ignoring several subtleties). Let the function being expanded be $G(\epsilon, X)$, where ϵ is the (small) asymptotic parameter and the $X = \{X_1, X_2, \ldots\}$ terms are other variables on which G depends. Let the bare (i.e., not resummed) asymptotic series be

$$G(\epsilon, X) = M_+(\epsilon, X) \exp\left\{\left(\frac{1}{\epsilon}\right)\phi_+(X)\right\} \sum_{r=0}^{\infty} \epsilon^r T_r(X), \tag{20}$$

where $T_0 = 1$ and the + indicates that the prefactor and exponent refer to the dominant exponential. As the order r increases, the terms $\epsilon^r T_r$ first get smaller and then diverge (typically factorially,[28] although the results do not depend on this[29]). Let the subdominant exponential (or the largest of these if there are several) have exponent $\phi_-(X)/\epsilon$, where

$$\mathrm{Re}\,\phi_-(X) < \mathrm{Re}\,\phi_+(X). \tag{21}$$

If now the series (equation 20) is optimally truncated, that is, summed to its least term r^* (typically, this increases as $1/\epsilon$), the remainder can be resummed and the resummation[26] has a remarkable universality:

$$G(\epsilon, X) \approx M_+(\epsilon, X)\exp\left\{\left(\frac{1}{\epsilon}\right)\phi_+(X)\right\}\sum_{r=0}^{r^*(\epsilon)}\epsilon^r T_r(X)$$
$$+ iS\{F(\epsilon, X)\}M_-(\epsilon, X)\exp\left\{\left(\frac{1}{\epsilon}\right)\phi_-(X)\right\}. \tag{22}$$

This includes the small exponential, which has been born from the divergent tail of the bare series (equation 20). Most important is the Stokes multiplier S, which describes the switching-on of the small exponential. S depends only on the disparity between the two exponents, denoted by F and called the singulant:

$$F(\epsilon, X) \equiv \left(\frac{1}{\epsilon}\right)[\phi_+(X) - \phi_-(X)]. \tag{23}$$

Stokes' phenomenon occurs when the large exponential maximally dominates the small one, that is, when F is real and positive; this occurs on Stokes lines (generally, hypersurfaces). The switching is smooth and is described by the universal function,[26]

$$S(F) = \left(\frac{1}{2}\right)\left[1 + \mathrm{erf}\left(\frac{\mathrm{Im}\,F}{\sqrt{2\,\mathrm{Re}\,F}}\right)\right], \tag{24}$$

where erf is the error function. Across the Stokes line, this rises from 0 to 1, with a width $\sqrt{\epsilon}$. The universality and compactness of this result is a consequence of the optimal truncation of the original series; if the series is not truncated at or near the least term r^*, these properties are lost.

A physical illustration of this general result is provided by the adiabatic two-state system defined in the INTRODUCTION. Here, the dominant exponential contains the dynamical phase associated with the amplitude for remaining in the initial adiabatic eigenstate and the small exponential describes the transition to the other state. Stokes' phenomenon plays a central part in the *history* of the transition amplitude,[30] that is, in describing how the quantum jump evolves as the amplitude increases from zero to its final exponentially small value (discussed in the third section). A Stokes line issues from the complex-time degeneracy τ^* responsible for the transition and crosses the real-time axis, at (say) $\tau = \tau_0$. It is at this time that the quantum jump can be said to happen.

If the transitions are considered to be between adiabatic states, that is, eigenstates of the instantaneous Hamiltonian, the transition history does not have the universal error function form (equation 24). The reason is that the adiabatic states are merely the first terms in the solution of equation 1 as an adiabatic power series in ϵ. If they are used as a basis to describe the transition, the amplitude increases from zero to $O(\epsilon)$ before falling to the final $O(\exp\{-1/\epsilon\})$, with complicated nonuniversal oscillations en route. However, if the adiabatic series for the solution of equation 1 is optimally truncated and the two solutions are employed as an "optimal superadiabatic" basis for describing the evolution, then the transition amplitude increases smoothly over a time of order $\sqrt{\epsilon}$ and in accordance with the error function (equation 24). Detailed analysis[30] shows that, for transitions from $|+\rangle$ to $|-\rangle$,

$$|\langle -(+\infty)|\psi(t)\rangle| = \left(\frac{1}{2}\right)\left[1 + \operatorname{erf}\left\{\frac{w(\tau)}{\sqrt{2\epsilon|w(\tau^*)|}}\right\}\right]\exp\left\{-\frac{|w(\tau^*)|}{\epsilon}\right\}, \tag{25}$$

where

$$w(\tau) \equiv \int_{\tau_0}^{\tau} d\tau' R(\tau'). \tag{26}$$

This surprising result shows that all adiabatic quantum transitions take place in the same way—that is, as an error function, independent of the form of the Hamiltonian curve—provided that the optimal superadiabatic basis is employed (which differs only slightly from the adiabatic one). Numerical calculations[31] confirm the correctness of the theoretical ideas, even when several degeneracies are involved,[32] which may coalesce.[33] There is a possibility that the Stokes smoothing could be observed directly in a spin experiment or an optical analogue: the spin would be measured at different times in a direction not along the "magnetic field" **R** (which would represent the adiabatic basis), but in a nearby direction, precisely specified by the theory,[30] representing the optimal basis.

DECOHERENCE AND THE QUANTUM ZENO EFFECT

Recent studies have identified two ways in which quantum systems are vulnerable to uncontrolled influences from their environment. The first is decoherence,[34,35] which is intended to explain why superpositions of macroscopically distinct states (as in Schrödinger's cat joke) are not observed. The mechanism is as follows: when the density matrix is represented in a basis of these macroscopically distinct states, averaging over the environmental forces or variables induces rapid vanishing of the off-diagonal elements. Because these elements represent quantum interference between the states, decoherence obliterates such interference, so the occupancies of the states can be described by probabilities rather than by amplitudes. The second way is the Zeno effect[36–39] (also known as the quantum watchdog or quantum watched pot), which is the slowing down, or complete inhibition, of quantum transitions that would occur in the isolated system, again as a result of action by the environment; it could explain, for example, the persistence of chiral molecules that, when isolated, would tunnel to their opposite-handed forms.

These phenomena are very different: in decoherence, the environment acts rapidly (relative to time scales associated with the isolated quantum system), whereas in the Zeno effect the environment induces slowing down. Therefore, it is worth knowing that both can be illustrated by a simple exactly solvable two-state model in the class defined in the INTRODUCTION (but with a different ϵ-dependence), and that is what I will present here. Neither the model nor its solution is new,[40–44] but I wish to emphasize how the two environmental effects are encompassed.

In equation 2, ψ_1 and ψ_2 represent the amplitudes for a quantum particle to inhabit each of the two wells, separated by a high barrier, of a double-bottomed potential. The wells could model, for example, the left- and right-handed forms of a chiral molecule. Forces within the isolated system are invariant under parity, that is, interchange of the two states (ignoring the small effect of weak interactions), so the eigenstates are even and odd superpositions of amplitudes localized in the wells. The energy separation of these eigenstates will be the small parameter ϵ, which is exponentially small in the height and width of the barrier and in $1/\hbar$. A state initially localized in one of the wells will tunnel resonantly between them, with an oscillation time of order $1/\epsilon$. The isolated system satisfies equation 1 with a different ϵ-dependence. The Hamiltonian (cf. equations 4 and 5) is

$$\mathbf{H}_0 = \mathbf{R}_0 \cdot \mathbf{S} = \left(\frac{1}{2}\right)\begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}, \tag{27}$$

where the suffix 0 denotes the isolated system.

A crude, but adequate, representation of the environment is through time-dependent random forces acting differently on the two wells (fuller treatments[45,46] include the environmental coordinates as dynamical variables). The random forces correspond to collisions, for example, with other molecules, or photons from the microwave background. In the asymptotic realm that we are interested in, where the two states are well separated by a high barrier, the environmental forces are large compared with ϵ and vary on time scales short compared with $1/\epsilon$. A Hamiltonian describing this situation is

$$\mathbf{H} = \mathbf{R}(t) \cdot \mathbf{S} = \left(\frac{1}{2}\right)\begin{pmatrix} f(t) & \epsilon \\ \epsilon & -f(t) \end{pmatrix}, \tag{28}$$

where $f(t)$ is white noise. The strength of the noise is normalized by

$$\overline{\left(\int_0^T dt f(t)\right)^2} = T/T_0. \tag{29}$$

Here and hereafter, the overbar denotes ensemble averaging and T_0 is a constant—the noise time—which can be interpreted as the time for the root-mean-square phase drift associated with $f(t)$ to grow to unity.

It is convenient to calculate the evolution of the ensemble-averaged density matrix,

$$\rho(t) \equiv \overline{|\psi(t)\rangle\langle\psi(t)|} = \begin{pmatrix} \overline{|\psi_1(t)|^2} & \overline{\psi_1^*(t)\psi_2(t)} \\ \overline{\psi_2^*(t)\psi_1(t)} & \overline{|\psi_2(t)|^2} \end{pmatrix}$$

$$\equiv \left(\frac{1}{2}\right) + \mathbf{S} \cdot r(t) = \left(\frac{1}{2}\right)\begin{pmatrix} 1 + z(t) & x(t) - iy(t) \\ x(t) + iy(t) & 1 - z(t) \end{pmatrix}, \tag{30}$$

conveniently expressed in terms of the Bloch vector,

$$r(t) = 2\overline{\langle\psi(t)|\mathbf{S}|\psi(t)\rangle}. \tag{31}$$

For a pure state ($\rho^2 = \rho$), $\boldsymbol{r}$ lies on the unit sphere. The north pole corresponds to all the amplitude in well 1 and the south pole to all the amplitude in well 2. The off-diagonal elements of ρ are described by the Bloch components, x and y. For any given $f(t)$ (noisy or otherwise), the unitarity of quantum evolution ensures that $\boldsymbol{r}(t)$ remains on the unit sphere, that is, pure. However, ensemble averaging, reflecting ignorance of the details of the environmental forces (or possibly time averaging—this is a subtle point), turns the pure state into a mixture and makes $\boldsymbol{r}(t)$ flow into the sphere. Then, it is the fate of any density matrix starting on the surface of the sphere to flow to the center, where there is equal probability of finding the particle in the wells and no interference between their amplitudes (for chiral molecules, this represents a racemic mixture of the two handednesses). Decoherence and the Zeno effect are features of the intervening evolution, now to be described.

The main steps in the determination of $\boldsymbol{r}(t)$ are explained in APPENDIX A. The form of the result depends on the parameter u defined by

$$u \equiv 4\epsilon T_0. \tag{32}$$

This is proportional to the ratio of the noise time to the quantum oscillation time. We are interested in small ϵ, that is, small u. For the initial condition of $\boldsymbol{r}(0) = \boldsymbol{r}_0$, the evolution is

$$x(t) = \exp(-t/2T_0),$$

$$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \frac{\exp\{-\alpha_+ t\}}{2\sqrt{1-u^2}} \begin{pmatrix} (1+\sqrt{1-u^2})y_0 + uz_0 \\ -uy_0 - (1-\sqrt{1-u^2})z_0 \end{pmatrix}$$

$$+ \frac{\exp\{-\alpha_- t\}}{2\sqrt{1-u^2}} \begin{pmatrix} -(1-\sqrt{1-u^2})y_0 - uz_0 \\ uy_0 + (1+\sqrt{1-u^2})z_0 \end{pmatrix}, \tag{33}$$

where

$$\alpha_\pm \equiv \left(\frac{1}{4T_0}\right)(1 \pm \sqrt{1-u^2}). \tag{34}$$

From the limits

$$\alpha_+ \to \frac{1}{2T_0}, \qquad \alpha_- \to 2\epsilon^2 T_0, \qquad \text{as} \qquad u \to 0, \tag{35}$$

we arrive at the following picture of the density matrix flow described by equation 34. On the scale of the noise time T_0, which is fast as compared with the quantum oscillation time $1/\epsilon$, the fast exponential, involving α_+, causes the Bloch vector to flow from the surface of the sphere towards the line $x = 0, y = -zu/(1 + \sqrt{1-u^2})$, which is close to the z-axis. This fast flow, in which the off-diagonal elements of ρ disappear, is decoherence. Thereafter, the flow down the z-axis towards $\boldsymbol{r} = 0$ is dominated by the slow exponential involving α_-, whose time scale is of order $1/(\epsilon^2 T_0)$ and thus long as compared with the quantum oscillation time. This slowing down of the quantum transitions (tunneling) is the quantum Zeno effect.

Loosely speaking, it is possible to regard the environment, acting through the random force $f(t)$, as monitoring, or measuring, the transitions of the particle between the wells, with the result that a pure state is rapidly reduced to a mixture (decoherence) and transitions are inhibited (Zeno). If r_0 is on the line $x = 0$, $y = -zu/(1 + \sqrt{1 - u^2})$, there is no decoherence, but only the Zeno effect in its simplest form; if r_0 is in the plane $z = -yu/(1 + \sqrt{1 - u^2})$ (close to the xy plane), there is no Zeno effect, but only decoherence in its simplest form.

It is clear that decoherence and the Zeno effect are phenomena that are asymptotically emergent as the parameter ϵ gets small and the quantum particle becomes increasingly vulnerable to its environment. Then, the three time scales, T_0 (noise and decoherence), $1/\epsilon$ (quantum oscillation), and $1/(\epsilon^2 T_0)$ (Zeno), separate.

The mechanism of the Zeno effect is that the fluctuating environment randomly shifts the wells up and down relative to each other. Then, only a small fraction of the time is available for tunneling because this can occur only when wells are within ϵ of each other and the eigenstates are shared between them (rather than being localized in an individual well). For this mechanism to operate, it is essential for the environment to fluctuate rapidly on the time scale of the transitions; otherwise, as the quantum adiabatic theorem implies, the amplitude will be completely transferred to the other well during each passage. Detailed analysis[47] of a single passage shows how the quantum and classical adiabatic theorems contradict each other in this situation and how this contradiction can be resolved; the same idea has been used in a discussion[48] of the persistence of chirality.

An interesting fact about the Zeno effect is that there is a sense in which its onset is independent of the form of the function $f(t)$, provided this is nonzero. I show in APPENDIX B that, if the initial amplitude is all in one well, the probability of finding the particle in the other well is always smaller with $f(t)$ than for the isolated system, provided t is less than the time π/ϵ for the amplitude to first switch wells in the isolated system.

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APPENDIX A

Exact Solution of the Two-State System Forced by White Noise

For convenience, we show here how the explicit solution of the model for decoherence and the quantum Zeno effect can be obtained by elementary means.

From equation 1, it follows that the Bloch vector $\boldsymbol{r}(t)$, defined by equation 31 without ensemble averaging, satisfies

$$\partial_t \boldsymbol{r}(t) = \mathbf{R}(t) \wedge \boldsymbol{r}(t), \tag{A1}$$

where

$$\mathbf{R}(t) = \{\epsilon, 0, f(t)\}. \tag{A2}$$

Regarding $\boldsymbol{r}$ as a column vector and introducing the matrix

$$\mathbf{M}(t) = \begin{pmatrix} 0 & -Z(t) & Y(t) \\ Z(t) & 0 & -X(t) \\ -Y(t) & X(t) & 0 \end{pmatrix}, \tag{A3}$$

we obtain the formal solution of equation A1 as the time-ordered product

$$\boldsymbol{r}(t) = T \exp\left\{\int_0^t \mathbf{M}(\tau)d\tau\right\}\boldsymbol{r}_0 \tag{A4}$$

for any $f(t)$. This must be ensemble-averaged. When $f(t)$ is white noise, averaging can be accomplished by dividing the time into small intervals Δ and using the fact that f is independent in different intervals. Then, the ensemble average is

$$\boldsymbol{r}(t) = (\overline{\exp\{\mathbf{M}(\tau)\Delta\}})^{t/\Delta}\boldsymbol{r}_0. \tag{A5}$$

To find the evolution of the density matrix, it is therefore necessary to exponentiate $\mathbf{M}$, average the result, and raise this to the power t/Δ. The exponential is

$$\exp\{\mathbf{M}\Delta\} = \left(\frac{1}{\epsilon^2 + f^2}\right)\left[\begin{pmatrix} 1 & 0 & \epsilon f \\ 0 & 0 & 0 \\ \epsilon f & 0 & f^2 \end{pmatrix} + \operatorname{Re}\exp\{i\Delta\sqrt{\epsilon^2 + f^2}\}\right.$$
$$\left.\times \begin{pmatrix} f^2 & if\sqrt{\epsilon^2 + f^2} & -\epsilon f \\ -if\sqrt{\epsilon^2 + f^2} & 1 + f^2 & i\epsilon\sqrt{\epsilon^2 + f^2} \\ -\epsilon f & -i\epsilon\sqrt{\epsilon^2 + f^2} & \epsilon^2 \end{pmatrix}\right]. \tag{A6}$$

Ensemble averaging eliminates the terms odd in f, and from the terms even in f it is necessary to retain only those contributions that will not vanish in the limit $\Delta \to 0$, after raising to the power t/Δ. To obtain these contributions, we need the variance of f over the interval Δ; from equation 29 (replacing the integral by a sum over intervals Δ), this is

$$\overline{f^2} = \frac{1}{T_0\Delta}. \tag{A7}$$

Then, the average of equation A6 is

$$\overline{\exp\{\mathbf{M}\Delta\}} \to \begin{pmatrix} 1-(\tfrac{1}{2})\Delta^2\overline{f^2} & 0 & 0 \\ 0 & 1-(\tfrac{1}{2})\Delta^2\overline{f^2} & -\epsilon\Delta \\ 0 & \epsilon\Delta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\left(\dfrac{\Delta}{2T_0}\right) & 0 & 0 \\ 0 & 1-\left(\dfrac{\Delta}{2T_0}\right) & -\epsilon\Delta \\ 0 & \epsilon\Delta & 1 \end{pmatrix}. \tag{A8}$$

Raising this to the power t/Δ gives the x evolution as

$$x(t) = \exp\left\{-\frac{t}{2T_0}\right\} \tag{A9}$$

and the $\{y, z\}$ evolution as

$$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \exp\{-t\mathbf{N}\}\begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \tag{A10}$$

where (cf. equation 3)

$$\mathbf{N} = \begin{pmatrix} 1/2T_0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} = \left(\frac{1}{4T_0}\right)\mathbf{1} + 2\epsilon\mathbf{S}_x + \left(\frac{1}{2T_0}\right)\mathbf{S}_z. \tag{A11}$$

This is easily exponentiated to give

$$\exp\{-t\mathbf{N}\} = \exp\left\{-\frac{t}{4T_0}\right\}\left[\cosh\left\{\left(\frac{t}{4T_0}\right)\sqrt{1-u^2}\right\}\mathbf{1} - \left(\frac{1}{\sqrt{1-u^2}}\right)\sinh\left\{\left(\frac{t}{4T_0}\right)\sqrt{1-u^2}\right\}\begin{pmatrix} 1 & u \\ -u & -1 \end{pmatrix}\right], \tag{A12}$$

where u is defined by equation 32.

Taken together, equations A9, A10, and A12 yield the claimed evolution (equation 33). It is interesting to examine the case where the noise is not large in comparison with ϵ; then, it is possible to have $u > 1$ and equation A12 is more conveniently written as

$$\exp\{-t\mathbf{N}\} = \exp\left\{-\frac{t}{4T_0}\right\}\left[\cos\left\{\left(\frac{t}{4T_0}\right)\sqrt{u^2-1}\right\}\mathbf{1} - \left(\frac{1}{\sqrt{u^2-1}}\right)\sin\left\{\left(\frac{t}{4T_0}\right)\sqrt{u^2-1}\right\}\begin{pmatrix} 1 & u \\ -u & -1 \end{pmatrix}\right]. \tag{A13}$$

This is the regime of quantum oscillations damped by tunneling friction.[43,46] The quantum Zeno effect corresponds to overdamping. Separating these two cases is critical damping ($u = 1$), for which

$$\exp\{-t\mathbf{N}\} = \exp\left\{-\frac{t}{4T_0}\right\}\left[\mathbf{1} - \left(\frac{t}{4T_0}\right)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right]. \tag{A14}$$

(I mention in passing that the matrix here is of the same degenerate non-Hermitian kind as discussed in the fourth section and this fact is responsible for the linear t-dependence.[11]) The limit of the isolated quantum system (no noise) is $T_0 \to \infty$, for which the evolution is unitary and consists of undamped quantum oscillations between the wells:

$$\exp\{-t\mathbf{N}\} = \cos\{\epsilon t\}\mathbf{1} - \sin\{\epsilon t\}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A15}$$

APPENDIX B

Initial Slowing Down of Transitions by Any Environment

For the two-state system (equations 1–4) with Hamiltonian (equation 28), the Schrödinger equation reduces, after the transformation

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} \equiv \begin{pmatrix} a_1(t)\exp\{(-\frac{1}{2})iw(t)\} \\ a_2(t)\exp\{(+\frac{1}{2})iw(t)\} \end{pmatrix}, \quad \text{where} \quad w(t) \equiv \int_0^t d\tau f(\tau), \tag{B1}$$

to the off-diagonal form

$$\partial_t\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} \equiv -i\left(\frac{\epsilon}{2}\right)\begin{pmatrix} a_2(t)\exp\{+iw(t)\} \\ a_1(t)\exp\{-iw(t)\} \end{pmatrix}. \tag{B2}$$

We start the system in the state 1, that is, $a_1(0) = 1$ and $a_2(0) = 0$. Next, we express the amplitudes a_1 and a_2 in terms of moduli $|a_1|$ and $|a_2|$ and phases χ_1 and χ_2, that is,

$$\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} \equiv \begin{pmatrix} |a_1(t)|\exp\{i\chi_1(t)\} \\ |a_2(t)|\exp\{i\chi_2(t)\} \end{pmatrix}. \tag{B3}$$

Thus, $|a_1|^2$ is the survival probability (for remaining in the initial state) and $|a_2| = \sqrt{(1 - |a_1|^2)}$.

From equation B3, it follows that

$$\partial_t|a_1(t)| = \left(\frac{\epsilon}{2}\right)|a_1(t)|\sin\{\phi(t)\}, \quad \text{where} \quad \phi(t) \equiv w(t) + \chi_2(t) - \chi_1(t). \tag{B4}$$

This has the solution

$$|a_1(t)|^2 = \cos^2\left\{\left(\frac{\epsilon}{2}\right)\int_0^t \sin[\phi(\tau)]d\tau\right\}. \tag{B5}$$

Now, if there is no noise, that is, $f = 0$, the exact solution of equation B2 is

$$a_1(t) = \cos\{(\tfrac{1}{2})\epsilon t\}, \qquad a_2(t) = -i\sin\{(\tfrac{1}{2})\epsilon t\} \qquad \text{if} \qquad f = 0, \tag{B6}$$

so

$$|a_1(t)| = \cos^2\{(\tfrac{1}{2})\epsilon t\} \qquad \text{if} \qquad f = 0, \tag{B7}$$

corresponding to $\phi = \pi/2$ if $f = 0$. For any nonzero $f(t)$, $\sin\phi < 1$, so $|a_1|^2$ decreases from unity more slowly than equation B7, at least until the first zero of equation B7, that is, $t = \pi/\epsilon$. This is the result stated at the end of the last section in the main text: for any environment $f(t)$, the survival probability exceeds that for the isolated system until its amplitude has switched wells for the first time.