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Integrals whose saddle-point expansions terminate

M V Berry

H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, United Kingdom

E-mail: asymptotico@bristol.ac.uk

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Abstract

The saddle-point expansion for integrals with integrand $\exp(-kf(x))$ is a series in powers of $1/k$. Usually this series diverges, but there is a family of exponent functions $f(x)$, defining a family of canonical integrals, for which the series terminates and the saddle-point expansion is exact. For this family, the transformation $x \to X$ such that $f(x) = X^2$ possesses a Jacobian that is a polynomial in X, whose coefficients parameterise the canonical integrals.

Keywords: steepest-descent, asymptotic, series

(Some figures may appear in colour only in the online journal)

1. Introduction

Formulations in mathematical physics have occasionally been reverse-engineered to provide new solutions, or new insights into old ones. Examples are: determining potentials, or refractive-index profiles, that are reflectionless [1]; finding time-dependent Hamiltonians that do not induce transitions between eigenstates of a related system [2–4]; and getting new solutions of the Schrödinger equation by fixing the Madelung–Bohm potential [5].

Here, in a similar spirit, I reverse-engineer the saddle-point approximation [6, 7], leading to a class of canonical integrals for which the technique is exact. In its simplest version, saddle-point integration concerns integrals of the form

$$I(k) = \sqrt{\frac{k}{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \ \exp(-kf(x)), \tag{1.1}$$

in which k is a large parameter and $f(x)$ a smooth function, increasing away quadratically from a minimum conveniently located at $x = 0$. The saddle-point (=steepest-descent =

stationary-phase with 45° rotated contour) approximation is central to theoretical physics: in statistical mechanics, where k is inverse temperature, and in wave mechanics, where k is imaginary and proportional to wavenumber or inverse Planck's constant. The technique generates an expansion of $I(k)$ as a series of inverse powers, whose coefficients involve derivatives of $f(x)$ at the origin. To save writing, we define the minimum value to be zero, and derivatives

$$f_n \equiv \partial_x^n f(x)_{x=0}, \quad f_0 = f_1 = 0, \quad f_2 > 0. \tag{1.2}$$

In the resulting formal series for $I(k)$,

$$I(k) = \sqrt{\frac{2}{f_2}} \sum_{n=0}^{\infty} \frac{I_n}{k^n}, \tag{1.3}$$

the coefficients I_n can be found by standard expansion methods; the first few are listed in table 1 in the appendix.

Usually, the series (1.3) is infinite, and factorially divergent in ways now well understood [8, 9]; the divergence depends on saddles and other singularities of $f(x)$ away from the real axis. But for special forms of $f(x)$ the series can terminate after $N + 1$ terms: $I_n = 0$ if $n > N$; then saddle-point integration is exact for all k. These special forms will be determined in section 3. This phenomenon, in which a series terminates for special values of a parameter, is familiar in the different context of the power series in the wavefunction for the quantum harmonic oscillator and the hydrogen atom): the series is a finite (Hermite, Laguerre) polynomial when the energy is one of the eigenvalues.

The saddle-point expansion for exponent functions $f(x)$ that increase quadratically away from their critical point is considered in section 2 because it is the simplest and most familiar situation. But the same procedure for calculating the canonical integrals applies, mutatis mutandis, when the increase of $f(x)$ from the saddle is any power of x. The generalisation is outlined in section 3, which also includes some concluding remarks.

2. Truncation formalism

To find the exponent functions $f(x)$ for which the series terminates, we start from the standard procedure in which the integration variable x is changed to X:

$$f(x) = X^2 \Rightarrow I(k) = \sqrt{\frac{k}{\pi}} \int_{-\infty}^{\infty} \mathrm{d}X \left(\frac{\mathrm{d}x}{\mathrm{d}X}\right) \exp\left(-kX^2\right). \tag{2.1}$$

The series is now determined by expanding the Jacobian dx/dX in powers of X. We require that this series terminates:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}X}\right) \equiv G^{(N)}(X) = \sum_{n=0}^{2N} G_n X^n. \tag{2.2}$$

This is a polynomial of degree $2N$, which must have no real zeros in order for the mapping $X(x)$ to be nonsingular. This restricts the coefficients; in particular $G_0 > 0$ and $G_{2N} > 0$. The result is a finite series for I, depending on k and the chosen coefficients G_n:

$$I\left(k; \{G_{2n}\}\right) = \sum_{n=0}^{N} \frac{G_{2n}}{k^n} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}} = G_0 + \frac{G_2}{2k} + \frac{3G_4}{4k^2} + \cdots \frac{\Gamma(2N)\, G_{2N}}{\Gamma(N)\, 2^{2N-1} k^N}. \tag{2.3}$$

Note that although the odd coefficients will contribute to $f(x)$, only the even coefficients G_{2n} contribute to the value of the integral.

The mapping $X(x)$ is found by integrating the Jacobian and then inverting the $2N+1$ degree polynomial:

$$x=\int_0^X \mathrm{d}X'\ G^{(N)}\left(X'\right)=\sum_{n=0}^{2N}\frac{G_n}{n+1}X^{n+1}\Rightarrow X=X^{(N)}(x)\,. \tag{2.4}$$

Of the $2N+1$ solutions, it is necessary to choose the one for which $X^{(N)}(x)\propto x$ at the origin, in order for the mapping to achieve its aim of standardising the behaviour at the saddle-point. For small and large x, the mapping is

$$\begin{aligned}X^{(N)}(x)&=\frac{x}{G_0}\left(1+O(x)\right)\\ X^{(N)}(x)&=\operatorname{sgn}(x)\left(\frac{(2N+1)\left|x\right|}{G_{2N}}\right)^{\frac{1}{2N+1}}\left(1+O\left(x^{-\frac{1}{2N+1}}\right)\right).\end{aligned} \tag{2.5}$$

From (2.1), the exponent function determining the integral whose series truncates after $N+1$ terms is

$$f^{(N)}(x)=\left(X^{(N)}(x)\right)^2. \tag{2.6}$$

For $N=0$, the procedure is trivial:

$$X^{(0)}(x,G_0)=\sqrt{f^{(0)}(x,G_0)}=\frac{x}{G_0}, \tag{2.7}$$

reproducing the fact that saddle-point integration is exact for Gaussian integrals. $N=1$ is non-trivial; solving the cubic equation (2.4) gives

$$\begin{aligned}X^{(1)}(x,G_0,G_1,G_2)&=\sqrt{f^{(1)}(x,G_0,G_1,G_2)}\\ &=-\frac{G_1}{2G_2}+\left(\frac{3}{2G_2}\right)^{1/3}\\ &\quad\times\left[\left(\sqrt{\xi^2+y^3}+\xi\right)^{1/3}-\left(\sqrt{\xi^2+y^3}-\xi\right)^{1/3}\right],\\ \text{where } y&\equiv\frac{4G_0G_2-G_1^2}{\left(12G_2^2\right)^{2/3}},\quad \xi=x+\frac{G_1\left(6G_0G_2-G_1^2\right)}{12G_2^2}.\end{aligned} \tag{2.8}$$

The condition for the mapping $X(x)$ to be nonsingular is $|G_1|<2\sqrt{(G_0G_2)}$, i.e. $y>0$. Although the integrand $\exp(-kf(x))$ depends on G_1, the integral does not:

$$I(k;G_0,G_2)=G_0+\frac{G_2}{2k}. \tag{2.9}$$

For $N>1$, determining the mapping, and hence the exponent functions $f(x)$, involves finding roots of quintic and higher polynomials. Although this cannot be carried out explicitly, the procedure is well-defined, and specifies the canonical integrals uniquely. The number of parameters can be reduced by 2, by the scalings

$$\alpha=\left(\frac{G_0}{G_{2N}}\right)^{\frac{1}{2N}},\quad X=\alpha X', \tag{2.10}$$

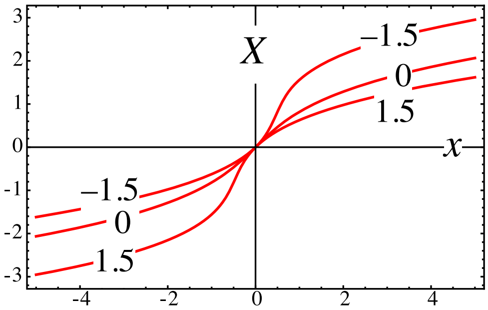

Figure 1. Mappings $X^{(1)}(x;1,G_1,1)$ (equation (2.8)) for the indicated values of G_1.

and the transformations

$$k'=\alpha k,\quad \left\{G'_0=1,\quad G'_{0<2n<2N}=\alpha^{2n}\frac{G_{2n}}{G_0},\quad G'_{2N}=1\right\}. \tag{2.11}$$

resulting in

$$I\left(k;\{G_{2n}\}\right)=G_0 I\left(k';G'_{2n}\right). \tag{2.12}$$

Therefore the integral depends on $2N-1$ essential parameters, in addition to k. For $N=1$ this leaves G_1 as the only parameter, and mappings illustrated in figure 1.

From the forms of $f(x)$, it is not obvious that the original saddle-point expansion is truncated, i.e. $I_{n>N}=0$. But it is, as will now be demonstrated for $N=1$. Expansion of the exponent function $f(x)$ given by (2.8) for $G_0=G_2=1$ gives the derivatives

$$\begin{aligned}&f^{(1)}(x,1,G_1,1)=\sum_2^\infty\frac{f_n x^n}{n!},\quad f_2=2,\ f_3=-6G_1,\ f_4=-16+30G_1^2,\\&\quad f_5=240G_1-210G_1^3,\ f_6=560-3360G_1^2+1890G_1^4,\ldots.\end{aligned} \tag{2.13}$$

Substitution into table 1 in the appendix gives $I_1=1/2$ (cf (2.9), independent of G_1 as claimed, and $I_2=I_3=0$.

3. Concluding remarks

In the method described here, an asymptotic technique that usually generates a divergent infinite series has been reverse-engineered to find integrals whose expansions terminate. The exponent functions $f(x)$ for which this is achieved are very special. An explicit test that would determine whether a given $f(x)$ is a member of this class would be useful but I have not been able to find one.

The procedure is easily generalised, in two ways. First to consider exponent functions $f(x)$ initially increasing as x^M, rather than x^2 as in the saddle-point method; second, to consider integrals over the positive real axis rather than the whole real line. This class of integrals, generalising (1.1), is

$$I(k)=\frac{k^{\frac{1}{M}}}{\Gamma\left(\frac{1}{M}\right)}\int_0^\infty \mathrm{d}x\ \exp\left(-kf(x)\right),\quad f_{n<M}(0)=0,\quad f_M>0. \tag{3.1}$$

The natural mapping is now $f(x) = X^M$. The same Jacobian occurs, and when represented by the same polynomial (2.2), with $2N$ replaced by N, it leads to the same mapping $X(x)$ (equation (2.4)) as before, also with $2N$ replaced by N, resulting in the exact truncated series, replacing (2.4),

$$I(k) = \sum_0^N \frac{\Gamma\left(\frac{n+1}{M}\right)}{\Gamma(1M)} \frac{G_n}{k^{nM}}. \tag{3.2}$$

The exponent function for which this truncated series exactly represents (3.1) is (2.6) with the exponent 2 replaced by M.

A technically more difficult generalisation would be to find truncated versions of uniform asymptotic expansions [10, 11], where saddle-points within the integration range coalesce as a parameter (additional to k) varies.

The parameter k in (1.1) need not be real. If it is imaginary, the analysis still applies, and gives truncated series for the method of stationary phase that is fundamental in diffraction theory [12] as the way to connect wave physics with ray optics or classical mechanics.

The exponent functions $f(x)$ in the canonical integrals possess branch points away from the real axis, at the values x mapped by the zeros of $G(X)$. In familiar asymptotics, such singularities contribute to the divergence of the expansion; for example, $f(x) = \sqrt{x^2 + y^2} - y$ in (1.1) generates the Bessel function $I(k) = 2\sqrt{k/\pi y}\ \exp(ky)\, K_1(ky)$ [13], whose large y expansion, generated by $G(X) = 2\left(X^2 + y\right)/\sqrt{X^2 + 2y}$, diverges. By contrast, the contributions from the branch-points must cancel for the canonical integrals considered here, because their series terminate.

In order to obtain terminated expansions, we have considered cases where the Jacobian $G(X)$ is a polynomial in X. But any function $G(X)$ for which the integral $\int_{-\infty}^{\infty} \mathrm{d}X\, G(X)\ \exp\left(-X^2\right)$ can be evaluated analytically leads to integrals that can be evaluated exactly. A class of such integrals corresponds to $G(X) = 1 + H(X)$, where $H(X)$ is any odd function. These integrals are not of the form (1.1) because their exponent functions $f(x)$ depend on k.

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Appendix. Saddle-point integration coefficients

With computer algebra, it is not difficult to generate the coefficients I_n in (1.3), in terms of the derivatives f_n. Table 1 shows the first few. The book [8] conveniently lists more: not only the coefficients I_n for $n \leqslant 4$, but also their dependence on the derivatives of a k-independent prefactor in the integrand, and the coefficients of half-integer powers of k that occur if the integration in (1.1) is restricted to the positive real axis (as in (3.1) with $N = 2$).

Table 1. Coefficients in the saddle-point expansion, from p 119 of [8].

n	I_n
0	1
1	$\frac{1}{24f_2^3}\left(5f_3^2-3f_2f_4\right)$
2	$\frac{1}{1152f_2^6}\left(385f_3^4-630f_2f_3^2f_4+105f_2^2f_4^2+168f_2^2f_3f_5-24f_2^3f_6\right)$
3	$\frac{1}{414\,720f_2^9}\left(425\,425f_3^6-1126\,125f_2f_3^4f_4+675\,675f_2^2f_3^2f_4^2\right.$ $-51975f_2^3f_4^3+360\,360f_2^2f_3^2f_5-249\,480f_2^3f_3f_4f_5+13608f_2^4f_5^2$ $\left.-\,83160f_2^3f_3^2f_6+22680f_2^4f_4f_6+12960f_2^4f_3f_7-1080f_2^5f_8\right)$

ORCID iDs

M V Berry https://orcid.org/0000-0001-7921-2468

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