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Roadmap

Roadmap on superoscillations

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Abstract
Superoscillations are band-limited functions with the counterintuitive property that they can vary arbitrarily faster than their fastest Fourier component, over arbitrarily long intervals. Modern studies originated in quantum theory, but there were anticipations in radar and optics. The mathematical understanding—still being explored—recognises that functions are extremely small where they superoscillate; this has implications for information theory. Applications to

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optical vortices, sub-wavelength microscopy and related areas of nanoscience are now moving from the theoretical and the demonstrative to the practical. This Roadmap surveys all these areas, providing background, current research, and anticipating future developments.

Keywords: imaging, optical beams, information theory

(Some figures may appear in colour only in the online journal)

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1. Faster than Fourier (p)revisited

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Past. The modern study of superoscillations was kick-started by Aharonov *et al* [1], who, in a still-unpublished 1991 preprint related to the then-new quantum weak measurements, envisaged a box containing only red light, which would emit gamma radiation when a window is opened. It was soon realised [2] that underlying this apparently paradoxical scenario was a deep mathematical phenomenon: a band-limited function ('red light') can vary arbitrarily faster than its fastest Fourier component ('gamma radiation'), over arbitrarily long intervals. Where superoscillations occur, the functions are exponentially weak (in the degree and extent of their 'faster than Fourier' variation), because the different Fourier components exhibit almost-perfect destructive interference. This weakness is the mechanism by which superoscillations evade the uncertainty principle (Fourier duality), because the principle is a relation between variances, and variances are insensitive to exponentially small values.

It is easy to create superoscillatory functions. Perhaps the simplest—certainly the most studied—is the periodic function

$$f(x) = \left(\cos\frac{x}{N} + \mathrm{i}a\sin\frac{x}{N}\right)^N, \qquad (1)$$

in which N is a large even integer and $a > 1$. This is periodic with period $N\pi$, and is band-limited, because when expanded in a Fourier series, the component oscillations are all of the form $\exp(\mathrm{i}k_n x)$ with $|k_n| \leqslant 1$. It is superoscillatory, because for $|x| < \sqrt{N}$ it can be approximated by $\exp(\mathrm{i}ax)$. Outside the interval $|x| < \sqrt{N}$, $f(x)$ first increases anti-Gaussianly and then rises to its enormous maximum value $|f(\pm N\pi/2)| = a^N$. The meaning of the parameters a and N is: a represents the degree of superoscillation in the region near $x = 0$, and N measures the extent of this superoscillatory region.

Superoscillations were anticipated in at least two other contexts. During World War II, research in microwave theory demonstrated that it was possible to design a radar antenna, consisting of many radiating elements in an arbitrarily small region, whose radiation pattern represents a beam whose angular width is arbitrarily small ('narrower than Rayleigh'). This 'superdirectivity' or 'supergain' is now understood in terms of superoscillations: in suitable variables, the radiation pattern is band-limited. But superdirectivity comes at a price, which has prevented extensive practical application: the individual elements must be driven very strongly, resulting in a near field that is exponentially more powerful than the narrow beam that reaches the far field.

Toraldo di Francia realised [3] that this microwave research has implications for optics, suggesting a lens with a focal spot small enough to enable superresolution (sub-wavelength) microscopy—performance beyond the Abbe resolution limit. The lens he designed required delicate fabrication, unavailable at that time. In 2000, the technology began to be available, and now the 'superoscillatory lens' (SOL) is being intensively developed, and the resulting superresolution microscopy is becoming practical [4]. An advantage of the SOL is that the microscopy is label-free, in contrast to STED microscopy which involves fluorescence, i.e. labelling. (STED also relies on superoscillation, in the sense that the depletion beam responsible for selective deactivation of fluorescence contains an optical vortex (see below), which can be arbitrarily narrow: there is no Abbe limit for dark light.)

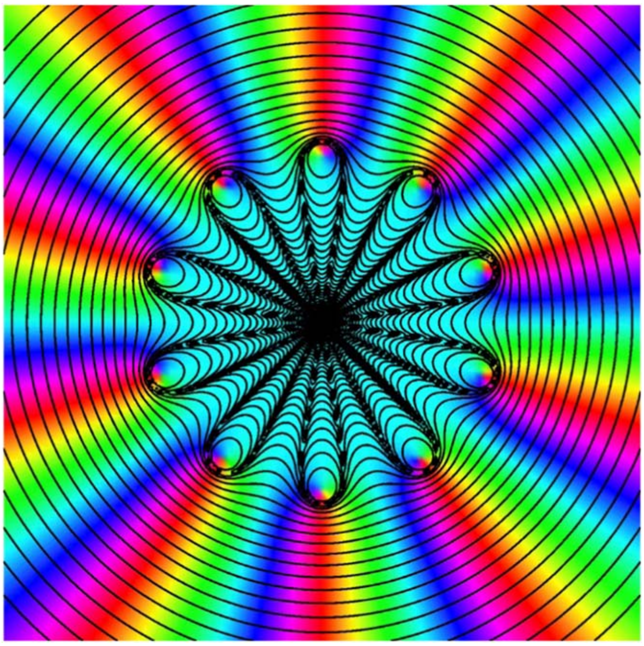

Figure 1. Superoscillatory fine detail in one square wavelength of the monochromatic wave $\psi = J_m(r)\exp(\mathrm{i}m\phi) + \varepsilon J_0(r)$ for $m = 1$, $\varepsilon = 10^{-7}$ over one square wavelength, The phase $\arg\psi$ is colour-coded, and the optical vortices are the ten points where all colours meet; superimposed are the lines of local wavevector $\mathrm{grad}(\arg\psi)$.

The second early context was phase singularities (=wave vortices, nodal points and lines, or wave dislocations), understood as topologically stable features of waves of all kinds [5]. Around a circuit of such a singular point P in the plane, the phase changes by 2π; so, close to P, the local phase gradient can be arbitrarily larger than any of the wavevectors in the Fourier superposition representing the wave. Therefore, all band-limited waves, in particular monochromatic ones, are superoscillatory near their phase singularities. This understanding emerged belatedly, in 2007; since then, research in superoscillations and phase singularities have merged. Dennis calculated [6] that superoscillations in waves are unexpectedly common: for random monochromatic light in the plane (e.g. speckle patterns), $1/3$ of the area is superoscillatory, with similar fractions for these 'natural superoscillations' in more dimensions. A monochromatic superoscillatory wave is illustrated in figures 1 and 2.

The large phase gradient that characterises superoscillations is alternatively described as the local wavevector. This illustrates the quantum 'weak measurement' scheme introduced by Aharonov and his colleagues, involving an operator

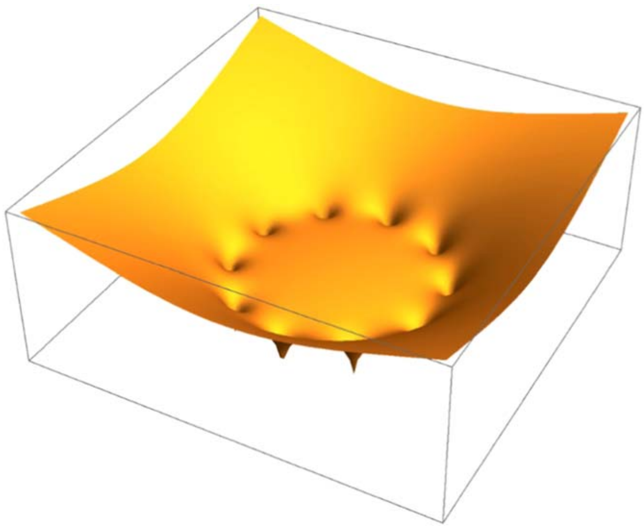

Figure 2. As figure 1, showing the intensity $\log|\psi|$.

with a finite spectrum, and pre- and postselected states, leading to a 'weak value' that lies outside the spectrum of the operator. In optics, the local wavevector is the weak value of the momentum operator, the preselected state is the wavefield and the postselected state represents position.

Present and future. The term superoscillation is slightly misleading, because not only fast oscillations but also fast-varying functions of any form can be reproduced band-limitedly. Examples are: the SOL, which generates a sub-wavelength spot; the possibility of reproducing Beethoven's Ninth Symphony with 1 Hz bandlimited signal [2]; and—a recent and extreme example—reproducing fractal functions to any desired accuracy.

A natural question arising in quantum applications is: if an initial state is a superoscillatory function of position, how long do the superoscillations persist under evolution according to the Schrödinger equation? An answer, obtained in 2006 [7], was that if the superoscillations extend over an interval $\sqrt{N}$ (where N is the large integer in (1)), the superoscillations persist for a time proportional to N. After this, they are destroyed by a rogue saddlepoint in the evolution integral. An implication is that for $N \to \infty$ the superoscillations would survive forever. The limit is strongly singular, and rigorous proof requires more sophisticated mathematics [8], being developed for a variety of evolutions.

In optics, the analogy between Schrödinger evolution in time and paraxial propagation in space suggested [7] that the $\sqrt{N}$ persistence might enable sub-wavelength microscopy without evanescent waves, because subwavelength detail in an object could reach a distant image plane. But paraxial propagation fails for superoscillatory light, and must be replaced by exact propagation according to the Helmholtz equation. It turns out, however, that there exists a class of initial waves, which can represent subwavelength detail, that propagate to repeat exactly at any chosen distance, and multiples of it.

Two fundamental obstructions to all extreme applications of superoscillations arise from their origin as a phenomenon of near-perfect destructive interference. The first is that such interference is inherently delicate, and survives only in the region (of size $\sqrt{N}$, mentioned earlier) where the Fourier amplitudes are phase-coherent. Outside this region, functions rapidly grow to values vastly greater than where they superoscillate. One implication is a difficulty for SOL microscopy: the dark ring around the narrow focal spot is surrounded by a ring of light that is exponentially brighter than the spot, threatening to burn vulnerable speciments.

Another consequence is that near-destructive interference is vulnerable to noise. This vulnerability has now been quantified for superoscillations contaminated by phase noise. As the noise increases through a tiny critical value, the phase coherence is destroyed; the local phase gradient decreases from its superoscillatory values and the tiny intensity increases, to magnitudes representative of the band-limited Fourier content.

These obstructions should be regarded positively, as challenges to the ingenuity of experimentalists. Particularly important is to develop ways of using superoscillations to go beyond the rather modest sub-wavelength resolutions currently obtainable in microscopy.

The original claim that gamma radiation can be released from a box containing red light has now been supported at the level of classical optics [9]. In a box (actually a tube) where superoscillatory red light is confined, a window is opened and closed as the superoscillations (moving with speed c) pass by. An analogy is with a curtain that is briefly opened in a lit room at night, releasing light allowing people in the darkness outside to see what is inside; but this is less straightforward when the light within is superoscillatory. Nevertheless, the exact solution of the relevant causal scattering problem shows that the superoscillations do escape into the far field as 'gamma radiation': the time-dependent window converts the fake frequencies in the superoscillations into genuine frequencies outside. An experiment to demonstrate this phenomenon would be worthwhile, though probably not easy.

At the quantum level, it is more difficult to understand how the escaping gamma photons get their energy, given that the box contains only red photons with much lower energy. This is the subject of a new paper by the original authors [10], arguing that the question opens basic issues concerning the interpretation of conservation laws in quantum physics.

Figure 19. Professors Sir Michael Berry, Yakir Aharonov and Nikolay Zheludev at the first international workshop on 'The Physics and Technology of Superoscillations' at the Institute of Physics, London, 16 October 2017. Used with permission.

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