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Geometric Phases

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1. Introduction

In one of its several scientific meanings, phase denotes the stages of a recurrent phenomenon or process. The phases of the moon describe its different shapes during each monthly cycle. Phase is represented mathematically by an angle; if new moon corresponds to 0˚, half-moon is 90˚, full moon 180˚, three-quarters (gibbous) moon is 270˚, and the next new moon is 360˚ which is equivalent to 0˚. In physics, phase describes the crests, troughs, and intermediate states of waves as they oscillate – waves of all kinds: quantum, electromagnetic, acoustic, elastic...

The familiar phase that accumulates as waves oscillate is dynamical. Geometric phases are additional contributions, arising, in their simplest form, when the conditions controlling a wave are slowly changed while it is rapidly oscillating, in such a way that the conditions return to their original form: a slow cycle superimposed on the fast ones.

For definiteness, consider a neutron in a magnetic field. The state of this quantum particle is a wave (a spinor) that depends on the strength and direction of the field. The wave's dynamical phase grows, at a rate depending on the strength of the field, and each 360˚ increase corresponds to one oscillation.

Now imagine that the field's direction (and possibly also its strength) is slowly changed, and then brought back to its original form. The wave has also returned to its original form, but has acquired a dynamical and a geometric phase.

The dynamical phase is large: proportional to the duration of the cycle; it is the accumulation of all the phases corresponding to the instantaneous frequencies of the changing field. The significance of the geometric phase, and the reason for its name, is that it does not depend on the speed of the field's change, or its duration, but only on the shape of the cycle: its geometry. For the neutron, the geometric phase is half the solid angle swept out by the magnetic field vector.

2. Quantum geometric phases

This slow-fast version of the geometric phase was discovered by Michael Berry in 1983, as a feature of the evolution of quantum states controlled by parameters that are slowly cycled. The parameters can be regarded as inhabiting an abstract space, in which the cycle is a closed curve – a circuit – whose shape determines the geometric phase.

During the evolution, each quantum state is part of a spectrum, determined by the instantaneous value of the parameters. For all points on the circuit, the energy of each state must be different from that of the others: the spectrum is non-degenerate. But degeneracies can occur for parameter values not on the circuit; these points can be regarded as sources of the geometric phase, abstract singularities analogous to magnetic monopoles, in the sense that associated with them is a field, called the 'phase 2-form' or 'curvature 2-form'. The geometric phase is the flux of this field through the circuit.

In the condensed-matter physics of electrons in two dimensions, the relevant abstract space is the Brillouin zone of the quantum Bloch states, whose parameters are the components of the quasi-momentum. The degeneracies of these states are points – sources of the curvature 2-form underlying the geometric phase – central to the understanding of important phenomena: the quantum Hall effect, topological insulators...

Geometric phases give an alternative interpretation of the AB effect, discovered by Yakir Aharonov and David Bohm in 1959. It is an example of quantum nonlocality: a quantum particle is influenced by magnetic fields inaccessible to it. The influence is equivalent to the geometric phase of an electron in a box that is transported round a line of magnetic flux.

3. Classical electromagnetic phases

Aspects of the phase were anticipated in classical electromagnetism, in several different ways. In 1941, Vassily Vladimirskii considered a linearly polarized light ray travelling along a gently curved path whose initial and final directions are the same. He predicted that the polarization at the end differs from that at the beginning: it has been rotated by an angle equal to the solid angle swept out by the ray direction. Much later, this polarization rotation was generated experimentally by light in a coiled optical fibre, and interpreted as a geometric phase of the circular polarization states into which the linear polarizations can be decomposed.

In the second electromagnetic anticipation, in 1956, Shivaramakrishnan Pancharatnam studied sequential changes in the polarization of a straight beam transmitted through a series of crystal plates. He discovered that such changes are non-transitive: if a beam starts in polarization 1, is transformed to polarization 2, then to polarization 3 and back to polarization 1, its phase has changed. Moreover the change is geometric: the sequence is a triangle 1231 on the 'Poincaré' sphere representing polarized light, and the phase is half the solid angle that the triangle subtended at the centre of the sphere. This situation is precisely analogous to the subclass of two-state quantum systems, of which the neutron is an example.

In the third anticipation, in 1975, Kenneth Budden and Martin Smith were studying the propagation of radio waves in the ionosphere, represented as a complex medium. In their formalism, they identified 'phase memory', formally identical to the dynamical phase in quantum physics, and 'additional phase memory', formally identical to what would be the quantum geometric phase for a closed circuit of radio waves.

4. Phases in classical dynamics

There are several counterparts of the geometric phase in classical dynamics. In 1985, John Hannay considered non-chaotic dynamics, where motion for fixed parameters is oscillatory and described by one or more angles. If the parameters are slowly cycled, each final angle differs from that accumulated when calculated from the instantaneous frequency; the discrepancy – the Hannay angle – depends on the geometry of the cycle. An example is the Foucault pendulum, swinging freely back-and-forth in a vertical plane at a point on the surface of the earth. The vertical rotates with the earth, and after a day the plane of the pendulum's swing has turned; the magnitude of the turn is the Hannay angle that would have been accumulated if the pendulum had been set into circular, rather than back-and-forth, motion.

Alfred Shapere and Frank Wilczek generalized the familiar phenomenon of the falling cat, which turns to land on its feet, even though its angular momentum remains zero because no torques act on it. They interpret this geometrically, in a theory of deformable bodies that can change their shapes, enabling them to turn in the absence of torques, by an amount determined by the circuit they execute in an abstract space of shapes. Similar geometry enables small organisms to swim in viscous fluids, as translational motion generated by cyclically changing their shapes.

5. Removing restrictions

The restriction to non-degenerate states was removed in 1984 by Frank Wilczek and Anthony Zee. They considered a cycle involving a set of N states that are degenerate for all points on the circuit. At the end of the cycle, the occupancies of the states can be different, as described by an $N \times N$ matrix rather than a phase.

The restriction to slow cycles was removed in a 1987 reformulation by Yakir Aharonov and Jeeva Anandan applicable to changes with any speed. They expressed the geometric phase for any quantum state that returns to its original state, in terms of a circuit in the 'projective quantum Hilbert space' of states, rather than the space of parameters driving the state.

6. Mathematical connections

Underlying geometric phases are several mathematical concepts. The 'global change without local change', by which the geometric phase appears at the end of the circuit, without appearing during the evolution according to the instantaneous frequencies, exemplifies a general geometric phenomenon: parallel transport. An example is an arrow, held horizontally and transported in a circuit over the curved surface of the earth; parallel transport corresponds to the arrow never being rotated about the vertical; nevertheless, it returns pointing in a different direction. Its turn is the solid angle subtended by the circuit at the centre of the earth; for the circuit from the north pole to the equator, then 90° west, then back to the north pole, the arrow's turn is 90° clockwise (viewed from above).

Related mathematics underlies the fact that the dynamics of quantum and other systems is determined by matrices, in which the physics is encoded in the eigenstates and where the control parameters appear as entries. When the parameters in a matrix are cycled, the eigenstates, when parallel-transported, acquire phases; in physical applications, these are the geometric phases.

This is one way to interpret the earliest geometric phase known to me, discovered by Humphrey Lloyd in 1831 while investigating the prediction of conical refraction by William Rowan Hamilton. A beam of light, incident on a suitably cut biaxially anisotropic crystal, would emerge as a bright cone, appearing on a screen as two closely split bright rings. As the figure illustrates, the emerging light is linearly polarized, with the direction turning by 180˚ around the rings. We know from Maxwell's equations that the polarizations of the light are eigenstates of a 2×2 matrix which depends on the crystal parameters and direction of the light, i.e. position around the rings. A circuit of the ring generates a geometric phase of 180˚ in the eigenstates, representing the observed polarization rotation.

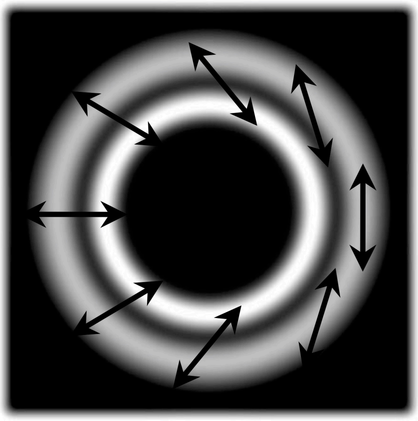

Figure. The first geometric phase, from conical refraction from a biaxial crystal: a half-turn of polarization directions of light, around the bright rings

7. Dynamics of parameters

In all the phenomena so far described, the parameters that generate the geometric phase are effectively external and controllable. But in some applications, the fact that the parameters are themselves physical variables is important; they are unavoidably reacted on by the systems they control. This is important in the quantum chemistry of molecules, whose heavy nuclei, moving slowly, can be regarded as controlling the light electrons, which move rapidly. In the Born-Oppenheimer approximation, the nuclei are regarded as instantaneously fixed, with configurations determining the quantum states of the electrons. For some nuclear configurations, the electron states are degenerate, and circuits around them generate geometric phases (usually 180˚). In 1959, Christopher Longuet-Higgins realised an implication of the requirement that the quantum state of the complete molecule should be single-valued: the nuclear quantum states acquire a compensating phase. This leads to observable effects in the energy spectrum of the molecule.

This insight was developed in 1979 by Alden Mead and Donald Truhlar. They interpreted what was later called the curvature 2-form of the fast electrons as a physical gauge potential influencing the slow nuclei. This potential is analogous to a magnetic field, and 'geometric magnetism' is now understood more widely, as the first correction to the slow dynamics, beyond the Born-Oppenheimer approximation or its analogues in other slow-fast systems.

This recognition that the parameters determining geometric phases are themselves dynamical has a fundamental implication: geometric phases are artefacts, resulting from our decision to consider a system as composed of interacting parts rather than a whole. For a molecule, direct solution of the Schrödinger equation for all the electrons and nuclei leads directly to the spectrum of quantum energy levels and states; no geometric phases need be considered. Nevertheless, almost all scientific practice relies on the fact that we do cannot study the entire universe; separation into 'system', in which we are interested, and 'environment', in whose details we are not, is inevitable. The geometric phases are thus reinstated, as one way in which the environment affects the system.

Further Reading

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Biography

Michael Berry is a theoretical physicist whose interests centre on connections between different levels of description, in particular geometric aspects of classical and quantum wave phenomena. He is at the University of Bristol, where he has been for more than twice as long as he has not.