

Hearing random matrices and random waves*

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Abstract. The eigenangles of random matrices in the three standard circular ensembles are rendered as sounds in several different ways. The different fluctuation properties of these ensembles can be heard, and distinguished from the two extreme cases, of angles that are distributed uniformly round the unit circle and those that are random and uncorrelated. Similarly, in Gaussian random superpositions of monochromatic plane waves in one, two and three dimensions, the dimensions can be distinguished in sounds created from one-dimensional sections.

 Online supplementary data available from stacks.iop.org/NJP/15/013026/mmedia

* This paper is dedicated to the memory of Richard E Crandall.

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1. Introduction

In recent decades it has become increasingly common to illustrate physics and mathematics with pictures. This complements abstract representations based on formulas, and reflects the growing recognition that visual representations can quickly convey ideas that may be cumbersome to express in words. A central role has been the development of friendly technologies (e.g. software) to create diagrams and graphs. In contrast to this exploitation of visual perception, the use of sound—representing abstract ideas in ways that can be heard—has lagged behind.

Our aim here is to explore the possibility of discriminating between different mathematical structures by creating corresponding sounds and listening to them. The sounds are presented as mp3 files. Hearing mathematics is not a new idea. A variety of functions has been rendered as sound, including fractals [1, 2], the ‘music of the primes’ and the Riemann zeta function [3–5]. The command `Play` in Mathematica [6] is a particularly flexible tool for creating a variety of musical and mathematical sounds. In what follows, the emphasis is on random functions. We present graphs and renderings of the associated sounds; these are samples from different ensembles, typical of many we explored, aimed at capturing the different fluctuation properties. The random functions are of two types.

The first (section 2) concerns hearing functions that represent points distributed around the unit circle as eigenangles of the principal circular ensembles of random-matrix theory [7]. These separate two extremes; the uniform distribution, in which the angles are equally spaced, and the Poisson distribution, in which they are random and uncorrelated. The angles are represented in three ways: as the zeros of circular secular polynomials (functional determinants) [7], in terms of the recently studied incompressibility function [8], and as the spacings between the angles. The different fluctuation properties of these arrangements of angles sound distinctly different. Our method for generating the circular ensembles is described in the [appendix](#); for related techniques see [9, 10].

The second (section 3) concerns monochromatic Gaussian random waves in one (1D), two (2D) and three dimensions (3D), that is, superpositions of many plane waves, all with the same wavelength, travelling in different directions. One-dimensional sections of such waves, regarded as functions of a single variable, have different (but band-limited) Fourier content. When rendered as sounds, the different dimensions can be distinguished.

There are many ways to render mathematical functions as sounds. The way we prefer here is to represent the functions as phase modulations of a carrier wave with a frequency in the range of comfortable human hearing. We think that this sounds clearer than representing them as sounds directly (and after considerable speeding up), as in a recent approach to rendering sounds related to arithmetic [5]. Undoubtedly, many other ways could be explored, and one of our aims in writing this paper is to encourage this. And as briefly discussed in the concluding remarks (section 4), there are other functions related to mathematics and physics whose rendering as sounds might be instructive.

This paper is frankly speculative. Whether hearing mathematics is useful, or merely a curiosity, remains to be seen—or rather, heard.

2. Hearing circular ensembles

We are interested in sonic rendering of the spectral fluctuations of large random matrices: $N \times N$ where $N \gg 1$. For the Gaussian ensembles, where the eigenvalues lie on the real line, the fluctuations are accompanied by a varying mean eigenvalue density (semicircle law). This could be eliminated by unfolding the spectrum [11] but is more conveniently accomplished by working with the circular ensembles: unitary (CUE), orthogonal (COE) and symplectic (CUE). For the sample matrices from these ensembles, the eigenvalues $\exp(i\theta_n)$ lie on the unit circle, involving N eigenangles which we order as follows:

$$0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_N < 2\pi. \quad (2.1)$$

In addition to the three ensembles, we will consider uniformly distributed angles, that is

$$\theta_n = \frac{2\pi}{N} \left(n - \frac{1}{2} \right) \quad (1 \leq n \leq N), \quad (2.2)$$

and Poisson-distributed angles, that is θ_N random on 0 to 2π .

2.1. Secular polynomials

For each matrix, the secular polynomial of the eigenvalues (functional determinant) is

$$\begin{aligned} D(\theta) &= \prod_{n=1}^N (\exp(i\theta) - \exp(i\theta_n)) \\ &= (2i)^N \prod_{n=1}^N \exp\left(\frac{1}{2}i(\theta - \theta_n)\right) \prod_{n=1}^N \sin\left(\frac{1}{2}(\theta - \theta_n)\right). \end{aligned}$$

The second equality suggests representing the spectrum by the real function

$$R(\theta) = 2^N \prod_{n=1}^N \sin\left(\frac{1}{2}(\theta - \theta_n)\right). \quad (2.4)$$

(This procedure is precisely analogous to replacing the Riemann zeta function on the critical line by $\zeta\left(\frac{1}{2} + it\right) = \exp(i\theta(t)) Z(t)$, with the phase $\theta(t)$ chosen so that $Z(t)$ is real [12].) Henceforth we assume that N is even, so $R(\theta)$ is periodic on the circle. For the uniform distribution (2.2),

$$R(\theta) = (-1)^{N-1} 2 \cos\left(\frac{1}{2}N\theta\right). \quad (2.5)$$

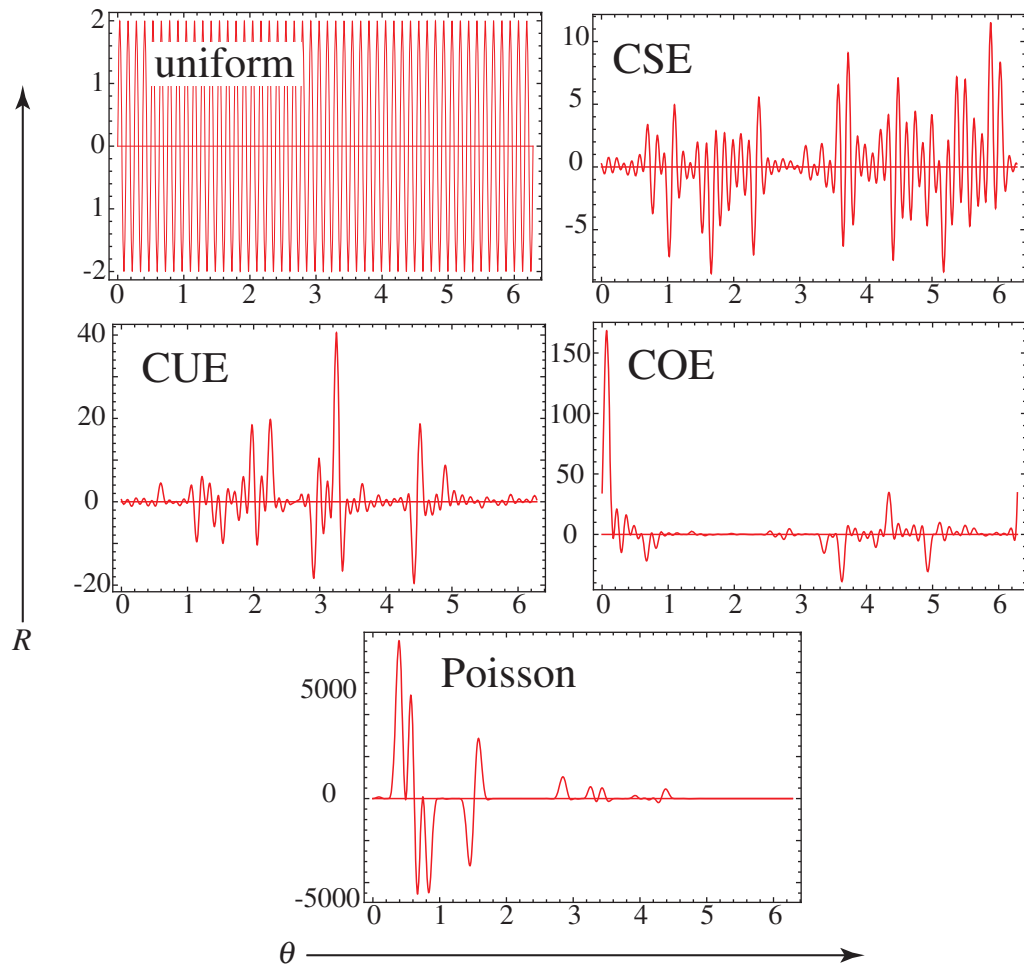


Figure 1. Secular polynomials for the cases indicated, calculated from (2.4) for $N = 100$.

Figure 1 shows sample polynomials $R(\theta)$ for the five cases: uniform, CSE, CUE, COE and Poisson, with the circular ensemble matrices generated as explained in the [appendix](#). The increasing fluctuations through the sequence—none at all for the uniform distribution, strongest for Poisson—are clear from visual examination, and from the numerical values on the ordinates of the graphs. The increasing fluctuations are associated with decreasing repulsion between the neighbouring angles.

The simplest way to render the functions $R(\theta)$ as sounds would be directly, as functions of time: $R(at)$, with $a = 4\pi\nu_0/N$ chosen so that the dominant frequency ν_0 , corresponding to the mean angle density $N/2\pi$, lies in the audible range. The uniform distribution would correspond to a pure tone with frequency ν_0 , and the other distributions by noisy signals with Fourier content centred on ν_0 . This requires very large N : for a sound with duration T to correspond to one complete cycle $\Delta\theta = 2\pi$, we would need $N = 2\nu_0T$, e.g. $N = 10\,000$ for $\nu_0 = 500$ Hz and $T = 10$ s. It is possible to diagonalize such large matrices, or employ the simpler though slightly imperfect shortcut of stringing in series the eigenangles of many smaller matrices.

We have tried this, and found it difficult to distinguish the noises corresponding to the different matrix ensembles.

Therefore we employ a different rendering, in which $R(\theta)$ is the phase modulation of a carrier wave with audible frequency ν_0 . Thus the function to be represented as sound is

$$f(t) = \sin\left(2\pi\nu_0 t + \Delta R\left(2\pi\frac{t}{T}\right)\right), \quad (2.6)$$

in which Δ is the amplitude of the modulation and T the duration of the sound, corresponding to one circuit encompassing all N angles. Five representative sound files are here included as sequence 2.1 (available from stacks.iop.org/NJP/15/013026/mmedia). When heard in the order uniform \rightarrow CSE \rightarrow CUE \rightarrow COE \rightarrow Poisson, the increasing fluctuations in this strange ‘random-matrix music’ are easy to hear. For the Poisson sound, the fluctuations are extreme, sometimes straying outside the audible range.

2.2. Incompressibility function

This is the circular version of the functions previously discussed in detail for the Riemann zeta function and the Gaussian ensembles. It is defined by [8]

$$Q(\theta) = \frac{\left(\sum_{n=1}^N \cot\left(\frac{1}{2}(\theta - \theta_n)\right)\right)^2}{\sum_{n=1}^N \csc^2\left(\frac{1}{2}(\theta - \theta_n)\right)}. \quad (2.7)$$

Among several interesting properties [8] are: $Q = 1$ at each nondegenerate eigenangle θ_n ; $Q = 0$ between each pair of neighbouring eigenangles; $Q = m$ at a degeneracy of m eigenangles; large values of Q correspond to large spectral fluctuations. For the uniform angle distribution (2.2),

$$Q(\theta) = \sin^2\left(\frac{1}{2}N\theta\right). \quad (2.8)$$

It turns out that the different fluctuations can be heard more clearly with $Q^2(\theta)$ than with $Q(\theta)$ itself, so that is what is presented here. We present six cases. In addition to the random-matrix and extreme functions, we include a stretch of the Riemann zeta function $Z(t)$ on the critical line [5, 12], the fluctuations of whose zeros are known to be the same as those of eigenvalues in the unitary ensemble (at least for not too distant zeros [13, 14]). Thus figure 2 shows $Q^2(\theta)$ for the sequence uniform \rightarrow CSE \rightarrow CUE \approx RiemannZeta \rightarrow COE \rightarrow Poisson. Again the progression of increasing fluctuations from the uniform angle distribution (most incompressible) through to Poisson (most compressible) is clearly visible, and the fluctuations of the zeta function and the CUE are similar.

And again the sounds representing the different cases are clearer when rendering as phase modulations, that is, with

$$f(t) = \sin\left(2\pi\nu_0 t + \Delta Q^2\left(2\pi\frac{t}{T}\right)\right). \quad (2.9)$$

Five representative sound files are included here as sequence 2.2 (available from stacks.iop.org/NJP/15/013026/mmedia). The increasing fluctuations in the sequence uniform \rightarrow CSE \rightarrow CUE \rightarrow RiemannZeta \rightarrow COE \rightarrow Poisson can be heard, with the difference that now, with $Q^2(q)$, rather than $R(\theta)$, the strong Poisson fluctuations remain in the audible range.

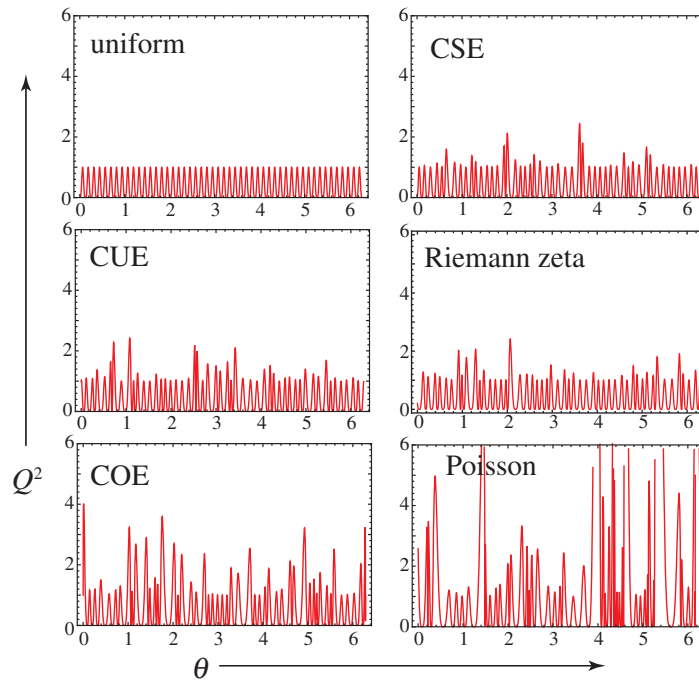


Figure 2. Incompressibility function Q^2 for the cases indicated, calculated from (2.7) for $N = 50$. For the Riemann zeta picture, Q^2 was calculated from the function $Z(t)$ [12], for 50 zeros starting from $t = 100\,000$.

2.3. Sounds of spacings

Normalized to unity, the $N - 1$ spacings between the eigenangles are the positive numbers

$$S_n = \frac{N}{2\pi} (\theta_{n+1} - \theta_n) \quad (1 \leq n \leq N - 1). \quad (2.10)$$

These can be rendered in sequence as musical pure tones with frequencies

$$\nu(S) = \nu_0 + S\Delta\nu, \quad (2.11)$$

and with each tone having duration τ . Thus the lowest note in this ‘spacings scale’ is ν_0 , and typical spacings $S \sim 1$ correspond to $\nu_0 + \Delta\nu$. The sound signal to be rendered is

$$f(t) = \sin(2\pi\nu S_{\text{Floor}(t/\tau)}t). \quad (2.12)$$

Four representative sound files are included here as sequence 2.3 (available from stacks.iop.org/NJP/15/013026/mmedia) (we do not give the file for the uniform distribution because this would consist of identical repeated notes with frequency $\nu_0 + \Delta\nu$). The greater fluctuations from CSE to CUE can just be discerned, but those from CUE to COE and from COE to Poisson are very distinct.

Again, other renderings are possible. For example, all the notes (2.11) could be played simultaneously, to generate the ‘spacings chord’ for each of the four cases. We find this less easy to interpret.

3. Hearing Gaussian random waves

The idea is to hear the difference between random waves in D dimensions, for $D = 1, 2, 3$. From many possibilities, we choose to hear monochromatic waves that are superpositions of N plane waves with wavevectors \mathbf{k} , such that $k = |\mathbf{k}| = 2\pi$ (i.e. unit spatial frequency), with \mathbf{k} isotropically distributed, and listen to the wave along a one-dimensional section (it will be the x -axis) at fixed time.

For $D = 1$, this is the trivial wave with $k_x = +1$ and -1 , the two contributions having equal amplitudes:

$$\psi_1(x) = \cos(2\pi x) \quad (3.1)$$

(figure 3(a)). The distribution of k (power spectrum) is simply the sum of two delta functions:

$$P_1(k_x) = \frac{1}{2} (\delta(k_x - 2\pi) + \delta(k_x + 2\pi)). \quad (3.2)$$

For $D = 2$, the wavevectors are

$$\mathbf{k}_n = \{k_{nx}, k_{ny}\} = 2\pi \{\cos \phi_n, \sin \phi_n\} \quad (1 \leq n \leq N), \quad (3.3)$$

with ϕ_n uniformly distributed from 0 to 2π . Thus, along the x -axis

$$\psi_2(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^N \cos(2\pi x \cos \phi_n + \gamma_n), \quad (3.4)$$

with random phases γ_n . Figure 3(b) shows a sample wave with $N = 50$. The distribution of k_x —that is, the power spectrum of this section of the wave—is

$$P_2(k_x) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \delta(k_x - 2\pi \cos \phi) = \frac{1}{\pi \sqrt{4\pi^2 - k_x^2}}. \quad (3.5)$$

For $D = 3$, the wavevectors are

$$\mathbf{k}_n = \{k_{nx}, k_{ny}, k_{nz}\} = 2\pi \{\sin \theta_n \cos \phi_n, \sin \theta_n \sin \phi_n, \cos \theta_n\}, \quad (1 \leq n \leq N), \quad (3.6)$$

with θ_n, ϕ_n uniformly distributed on the unit sphere. Thus, along the x -axis

$$\psi_3(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^N \cos(2\pi x \sin \theta_n \cos \phi_n + \gamma_n), \quad (3.7)$$

with random phases γ_n . Figure 3(c) shows a sample wave with $N = 50$. The power spectrum of k_x for this case is

$$P_3(k_x) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \delta(k_x - 2\pi \sin \theta \cos \phi) = \frac{1}{2} \Theta(2\pi - |k_x|), \quad (3.8)$$

in which Θ denotes the unit step—that is, the spectrum is flat in this case.

All three spectra are band-limited, with the greatest spatial frequency being unity, i.e. $|k_x| \leq 2\pi$. Within this range, the spread of wavenumbers increases from 1D to 2D to 3D: in

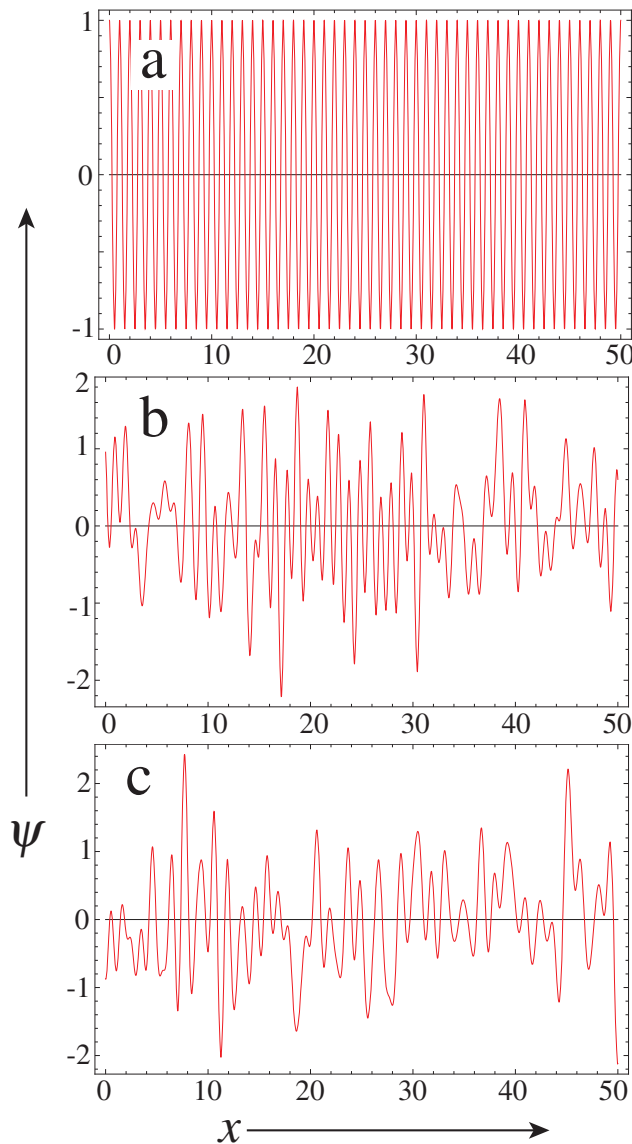


Figure 3. (a) One-dimensional wave (3.1) with spatial frequency unity; one-dimensional section (3.4) of Gaussian random wave in two dimensions, constructed from $N = 50$ plane waves; (c) as (b), for a wave in three dimensions.

1D, all Fourier content is confined to the ends of the spectrum; in 2D, the power spectrum exists for all k_x in the range but there are singularities at the ends; and in 3D the spectrum is flat. This is clear from (3.2), (3.5) and (3.8), and also from figure 3 (note that figure 3(c) there are more low frequencies).

As with the random-matrix spectra, the clearest way to render the different cases is as phase modulation. In this case, suitable functions are

$$f_i(t) = \sin\left(2\pi\nu_0 t + \Delta\psi_i\left(\frac{t}{\tau}\right)\right) \quad (i = 1, 2, 3, \quad 0 \leq t \leq T). \quad (3.9)$$

The three different band-limited spectra generate different sounds, included here as sequence 3 (available from stacks.iop.org/NJP/15/013026/mmedia), with the modulation getting more irregular in the sequence 1D \rightarrow 2D \rightarrow 3D.

4. Concluding remarks

As we have already remarked several times, the renderings presented here are far from unique. We have explored several other ways to represent random functions, and more can be envisaged. By more elaborate processing, it is likely that the sounds can be made more musical, and possibly more informative as well—for example by incorporating rhythm, harmony, and intensity as well as phase modulation. What we have presented is simply a first step.

One aspect not considered here is the possibility of hearing the complementary aspects of a function and its Fourier transform. One of us has explored this recently [5] in the context of the counting function for the prime number fluctuations, whose Fourier ‘harmonies’ are the zeros of the Riemann zeta function. This revealed an acoustic complementarity: it is possible to hear the singularities in the counting function, or the harmonies, but not both together. Closely related is the complementarity between the quantum energy-level spectrum of a classically chaotic quantum system (for example as manifested in the spectral determinant) and the ‘spectrum’ of lengths of the classical periodic orbits [15, 16]. It would be interesting to explore whether this can be heard by rendering the quantum spectrum as sound.

Many other aspects could be explored. One is rendering eigenfunctions; in the case of disordered systems, it would be especially interesting to try to hear the difference between localized and extended states, and the fractal structure of states at criticality (e.g. in the metal–insulator transition). Another is superoscillations, that is band-limited functions that oscillate faster than their fastest Fourier components [17–19]. Finally, it would be worth studying connections between these ways of representing mathematical functions as sounds and the cognitive psychology of acoustic perception (see [20] for a recent comprehensive account of this and other aspects of hearing).

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Appendix. Generating the circular ensemble matrices

We used a variant of known methods [9, 10]. The first step is to generate $N \times N$ Hermitian matrices H from the Gaussian unitary ensemble, that is

$$H_{ij} = \frac{1}{2}(a_{ij} + ib_{ij} + a_{ji} - ib_{ji}) \quad (1 \leq i, j \leq N), \quad (\text{A.1})$$

with a_{ij} and b_{ij} being independent real random numbers, identically Gauss-distributed.

The next step is to use the orthonormal eigenvectors $u_j^{(i)}$ of each H to generate a CUE matrix C_U ; here i labels the eigenvectors and j their components. Incorporating random phases γ_i , the CUE matrices are

$$C_{U,ij} = \exp(i\gamma_i) u_j^{(i)}. \quad (\text{A.2})$$

The random phases eliminate any bias that may be introduced by software routines for generating eigenvectors, which can skew the circular ensembles away from the desired CUE, COE and CSE. For example, Mathematica's routine makes the N th component of all the eigenvectors real and positive, with the consequence that the COE spectrum is always contaminated and distorted by a zero eigenangle (we leave the proof as an exercise).

In the next step, we use each C_U and its transpose to generate a COE matrix C_O from

$$C_O = C_U^T C_U. \quad (\text{A.3})$$

Finally, the CSE ensemble of matrices C_S is generated from the CUE matrices with dimension $2N \times 2N$ by

$$C_S = C_U^R C_U, \quad (\text{A.4})$$

where

$$C_U^R = Z C_U^T Z, \quad (\text{A.5})$$

in which Z is a $2N \times 2N$ unit symplectic matrix, calculated from the outer product of the $N \times N$ identity matrix with the unit antisymmetric 2×2 matrix (in Mathematica, this can be accomplished with the command `KroneckerProduct`). The resulting $2N \times 2N$ matrices have N distinct eigenangles, each doubly degenerate (Kramers degeneracy).

We checked the ensembles thus created in two ways. Firstly, by confirming that the eigenvalues are uniformly distributed on the unit circle; this is not obvious because, as is known [9], superficially similar algorithms can give non-uniform eigenangle densities. Secondly, by confirming that the eigenangle nearest-neighbour spacings distributions $P(S)$ correspond to those of the CUE, COE and CSE, with their very different forms of eigenangle repulsions. For the COE, our computations were sufficiently accurate to distinguish the exact large N form for $P(S)$ (tabulated in [21]) from its Wigner approximation $32S^2/\pi^2 \exp(-4S^2/\pi)$.

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