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FAST TRACK COMMUNICATION

Hearing the music of the primes: auditory complementarity and the siren song of zeta

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Abstract

A counting function for the primes can be rendered as a sound signal whose harmonies, spanning the gamut of musical notes, are the Riemann zeros. But the individual primes cannot be discriminated as singularities in this 'music', because the intervals between them are too short. Conversely, if the prime singularities are detected as a series of clicks, the Riemann zeros correspond to frequencies too low to be heard. The sound generated by the Riemann zeta function itself is very different: a rising siren howl, which can be understood in detail from the Riemann–Siegel formula.

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1. Introduction

Riemann [1, 2] showed that the fluctuations of the prime numbers about their mean density can be described by a Fourier-like series of oscillations, whose frequencies are given by the celebrated complex zeros of his zeta function. The implied analogy with music has often been noted in lectures, in the title of a popular book [3], and by Bombieri [4]:

> 'To me, that the distribution of prime numbers can be so accurately represented in a harmonic analysis is absolutely amazing and incredibly beautiful. It tells of an arcane music and a secret harmony composed by the prime numbers.'

As described later, the 'music' is easy to create on a computer; it can be heard online [5], accompanying visual depictions of the underlying signal.

My purpose here is to explore this idea further, to see how far the primes, and the associated Riemann zeta function, can be represented by a sound signal incorporating the gamut of musical notes. The simplest prime signal is described in section 2. The implications for hearing the music are analyzed in section 3. The main result is that there is a sense in which the Fourier and prime representations are complementary: when the music of the primes is synthesized as a superposition of Riemann zeros, harmonies can be heard but individual primes are inaudible; and if the signal is slowed so as to hear the primes, the harmonies are below the threshold of hearing. The signal has fractal aspects, discussed in section 4. The very different sound generated by the Riemann zeta function is analyzed in section 5.

2. Prime counting signal

The simplest prime counting function might seem to be the staircase: $\pi(x) =$ number of primes less than x. But it is well known [1] that the connection with the Riemann zeros is simpler with the counting function for prime powers p^n, using the convenient weighting $\log p$, namely Riemann's psi function

$$\psi(x) = \sum_{p^n<x} \log p \quad (p = 2, 3, 5, 7, 11\ldots, \quad n = 1, 2, 3\ldots). \tag{2.1}$$

This can be decomposed into its smooth and fluctuating parts:

$$\psi(x) = \psi_{\mathrm{sm}}(x) + \psi_{\mathrm{fluct}}(x). \tag{2.2}$$

The smooth part, close to linear, is

$$\psi_{\mathrm{sm}}(x) = x - \log(2\pi) - \tfrac{1}{2}\log(1 - x^{-2}). \tag{2.3}$$

Here the emphasis is on the fluctuating part, defined exactly in terms of the complex Riemann zeros ρ by [1]

$$\psi_{\mathrm{fluct}}(x) = -\sum_{n=1}^{\infty} \frac{x^\rho}{\rho} \quad \left(\rho = \frac{1}{2} \pm \mathrm{i}t_n, \quad n = 1, 2, 3\ldots\right). \tag{2.4}$$

The numbers t_n are the heights of the zeros; if the Riemann hypothesis (RH) is true, all the complex zeros lie on the critical line $\mathrm{Re}\,\rho = 1/2$, so all t_n are real.

Assuming RH, each term in (2.4) represents an oscillation whose phase is $\arg(x^\rho) = t_n \log x$. The corresponding angular frequency $\mathrm{d(phase)}/\mathrm{d}x = t_n/x$ depends on x. This is awkward for a sound signal, in which it is desirable for each Riemann zero to represent a pure tone. Therefore we change to a time variable τ proportional to $\log x$:

$$x = \exp(a\tau), \tag{2.5}$$

involving a scaling constant a. It is convenient to remove the factor $\sqrt{x}$ in ψ_{fluct}, associated with $\mathrm{Re}\,\rho = 1/2$, and thus define the *sound signal representing the primes* as

$$S(\tau) = \exp\left(-\tfrac{1}{2}a\tau\right)\psi_{\mathrm{fluct}}(\exp(a\tau)) = -2\mathrm{Re}\sum_{n=1}^{\infty} \frac{\exp(2\pi \mathrm{i}\nu_n\tau)}{\tfrac{1}{2} + \mathrm{i}t_n}. \tag{2.6}$$

Assuming RH, this function has a discrete spectrum, whose frequencies—the harmonies of the primes—are

$$\nu_n = \frac{at_n}{2\pi}. \tag{2.7}$$

The amplitudes are $1/\sqrt{t_n^2 + \frac{1}{4}}$. If RH would be false, some of the ν_n would be complex, and the spectrum would not be purely discrete. Extending the definition of 'music' to include any signal with a discrete spectrum, this enables a nontechnical statement of RH [6]: the primes contain music.

3. Complementarity of primes and Riemann zeros

To implement $S(\tau)$ as music, we choose the frequency ν_1, associated with the lowest Riemann zero, as the lowest note on the piano keyboard, and replace the sum (2.6) by the truncated version

$$S_N(\tau) = -2\mathrm{Re}\sum_{n=1}^{N}\frac{\exp(2\pi \mathrm{i}\nu_n\tau)}{\frac{1}{2}+\mathrm{i}t_n}, \tag{3.1}$$

including N zeros, where t_N is associated with the highest note on the piano keyboard. Thus, with τ measured in seconds,

$$\begin{aligned}\nu_1 &= 27.5\,\mathrm{Hz}\,(\text{musical note A0})\\ \nu_N &= 4186.01\,\mathrm{Hz}\,(\text{musical note C8}).\end{aligned} \tag{3.2}$$

The scaling constant a in (2.5), and the index N, now follow from (2.7):

$$a = 12.224, \quad t_N = \frac{\nu_N}{\nu_1}t_1 = 2151.57 \Rightarrow N = 1657. \tag{3.3}$$

This gives the 'Riemann scale' as the set of frequencies $\nu_1, \nu_2, \ldots, \nu_N$.

It is easy to create a computer program to enable the scale (2.7) and the 'music' (3.1) to be heard. One such program accompanies this paper as supplementary material (available from http://stacks.iop.org/JPhysA/45/382001/mmedia), along with sound clips of the scale and signal. A version incorporating 100 zeros has been posted online [5]. Interpreting the sound as music requires some imagination: although the low zeros can be discerned as ghostly growls, the signal sounds like noise, for reasons explained in section 4. Alternative implementations are easy to explore. For example, starting at a higher note makes the low harmonics stand out more clearly, at the price having fewer notes in the musical gamut: for ν_1 = A2 = 110 Hz, corresponding to $a = 48.896$, $N = 297$.

From its definition in terms of $\psi(x)$, the signal (2.6) has singularities: discontinuities corresponding to the prime powers p^n, at times $\tau = (n/a)\log p$. These singularities cannot be heard in the prime music as defined here. To understand why, note first that for very short times it is possible to discriminate individual prime powers in the truncated series (3.1), even with far fewer zeros N, as illustrated in figure 1(*a*) for $0 < \tau < 0.2$ s. But this association soon dissolves, as illustrated in figures 1(*b*)–(*d*). The reason is that to resolve detail in $\psi(x)$ or $S(\tau)$ on a scale $\Delta x = \log x$ corresponding to the spacing between primes at x, the synthesis must include at least the first M Riemann zeros, given by the phase change

$$\Delta(t_M \log x) = t_M\frac{\Delta x}{x} = \frac{t_M \log x}{x} = 2\pi, \text{ i.e. } t_M = 2\pi\frac{x}{\log x}. \tag{3.4}$$

It now follows from the known counting function of the zeros [1], namely $N(t) \approx (t/2\pi)\log(t/2\pi e)$, that

$$M \sim x = \exp(a\tau). \tag{3.5}$$

This implies that discrimination of the jumps at individual prime powers, even for times inaudibly close to the start of the signal, would require an unfeasibly large number of Riemann zeros: for $\tau = 1$ s, $M \sim 204\,000$; and for $\tau = 10$ s, $M \sim 1.2 \times 10^{53}$. With $M = N = 1657$ zeros, corresponding to the highest piano note, primes cease to be discriminated for $\tau > \log(a)/N \sim 0.6$ s. Increasing the gamut to include the full range of human hearing (optimistically, up to 20 kHz) hardly helps.

Even if the number of Riemann zeros is increased vastly beyond the audible range, so that individual primes could be discriminated in principle, they would still be inaudible, because

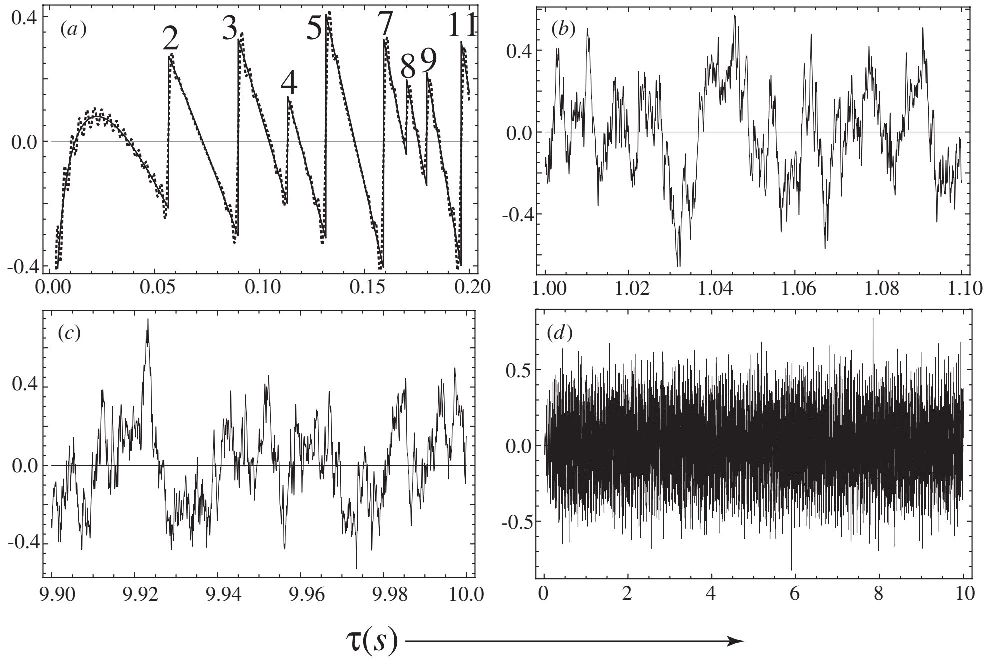

Figure 1. (*a*) Comparison of the truncated signal $S_{50}(\tau)$ (equation (3.1)) (dotted curve) with the exact $S(\tau)$ (equation (2.6)) (full curve), for $0 < \tau < 0.2$ s; the numbers at the jumps indicate the corresponding prime powers in $\psi(x)$. (*b*) Truncated signal $S_{1657}(\tau)$, for 1.0 s $< \tau <$ 1.1 s. (*c*) Truncated signal $S_{1657}(\tau)$, for 9.9 s $< \tau <$ 10.0 s. (*d*) Truncated signal $S_{1657}(\tau)$, for $0 < \tau < 10$ s.

the time intervals $\Delta\tau$ between their singularities in the signal are too short to be heard. For the primes at time τ,

$$\Delta\tau = \frac{\Delta x}{ax} = \frac{\log x}{ax} = \tau \exp(-a\tau). \tag{3.6}$$

Even for the rather short time $\tau = 1$ s, $\Delta\tau \sim 5$ μs; and for $\tau = 10$ s, $\Delta\tau \sim 8 \times 10^{-53}$ s. As figures 1(*b*)–(*d*) illustrate, at such times the singularities at individual prime powers are invisible. Instead, the graphs look fractal, an aspect to be discussed in the next section.

Of course, it is possible to hear the individual prime power singularities in $S(\tau)$ as a series of clicks, simply by slowing the signal, that is, by reducing the scaling a. For example, to hear the prime powers 2, 3, 4, 5, 7, 8, 9 in the time interval $0 < \tau < 10$ s, it is necessary to take $a = \log(10)/10 = 0.2306$. But then the lowest Riemann zero t_1 contributes with frequency $\nu_1 = 0.517$ Hz—far below the threshold of hearing—and the lowest piano note frequency A0 = 25.5 Hz is reached only at the zero t_{413}.

So, it is possible to hear the prime power clicks, or the Riemann zero harmonies, but not both at the same time. This complementarity reflects the different scales at which the individual Riemann zeros and the individual prime powers occur in the signal $S(\tau)$. Each Riemann zero describes an oscillation with period T_n which is independent of τ: $T_n = 1/\nu_n = 2\pi/at_n$ (cf (2.7)). The prime powers—jumps in $S(\tau)$—correspond to the much smaller exponential scale (3.6). The period T_n represents a scale of oscillatory clustering of prime powers in which each cluster contains $T_n/\Delta\tau = (2\pi/at_n)\exp(a\tau)/\tau$ prime powers; the number increases exponentially as τ increases.

On the spectral interpretation of the Riemann zeros as energy levels (eigenvalues) of a classically chaotic dynamical system [6–8], the prime powers represent repetitions of periodic orbits. $S(\tau)$ in (2.6) is a trace formula, with singularities at the prime powers (periodic orbits) represented as oscillations determined by the Riemann zeros (energy levels); this is complementary to the spectral density trace formula [9–11] in which singularities at the energy levels (Riemann zeros) are represented as oscillations determined by the periodic orbits (primes). Discriminating individual primes in the signal $S(\tau)$ synthesized from the Riemann zeros is analogous to 'inverse quantum chaology' [12], in which the periods of classical periodic orbits are determined from the quantum energy levels. It would be interesting to know if the auditory complementarity identified here extends to 'music' representing classically chaotic dynamical systems, in which the harmonies are the quantum energy levels.

4. Fractal structure of the prime signal

On the finest scale (3.6), exponentially small as τ increases, $S(\tau)$ consists of singularities at prime powers, illustrated in figure 1(*a*). On coarser scales, the graph of $S(\tau)$ looks fractal (figures 1(*b*)–(*d*)). The associated dimension D can be estimated as follows. Writing (2.6) as

$$S(\tau) = \text{constant} \times \text{Re} \sum_n \frac{\exp(\mathrm{i}\omega_n \tau)}{\left(\frac{1}{4} + \mathrm{i}\omega_n/a\right)} \tag{4.1}$$

with $\omega_n = t_n/a$, the corresponding power spectrum is

$$P(\omega) \propto \sum_n \frac{\delta(\omega - \omega_n)}{\omega_n^2 + \frac{1}{4}a^2} \sim \frac{1}{\omega^2}\frac{\mathrm{d}n(\omega)}{\mathrm{d}\omega} = \frac{1}{\omega^2}\log\left(\frac{a\omega}{2\pi}\right), \tag{4.2}$$

where the last equality incorporates the asymptotic density of the t_n.

Now we use the result that the graph of a Fourier series $S(\tau)$ with uncorrelated phases and power-law power spectrum $P(\omega) \propto \omega^{-\mu}$ with $1 < \mu < 3$ is a continuous but nondifferentiable curve with fractal dimension $D = (5-\mu)/2$ [13, 14]. Up to logarithms, (4.2) is a power-law with $\mu = 2$, giving $D = 3/2$. The observation that this is the dimension of the graph of Brownian motion in one space dimension quantifies the pseudo-randomness of the prime powers on coarse scales and explains why the music sounds like noise. Visually, comparison of figures 1(*b*) and (*c*) with other fractal curves [14] with a variety of dimensions is consistent with the value $D = 3/2$. A more refined measure-theoretic analysis, involving concepts beyond the fractal dimension and incorporating the logarithm, would probably indicate additional weak scale-dependent roughness in the graph of $S(\tau)$.

5. The song of zeta

The natural way to render the zeta function $\zeta(s)$ on the critical line $s = 1/2 + \mathrm{i}t$ as a sound is by the scaled version $Z(at)$ of the real function [1]

$$Z(t) = \exp(\mathrm{i}\theta(t))\zeta\left(\tfrac{1}{2} + \mathrm{i}t\right), \tag{5.1}$$

in which the phase $\theta(t)$ is

$$\theta(t) = \text{Im}\log\Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}\mathrm{i}t\right) - \tfrac{1}{2}t\log\pi. \tag{5.2}$$

The sound generated by $Z(at)$, as described in the supplementary material (available from http://stacks.iop.org/JPhysA/45/382001/mmedia), is very different from the music of the primes, in ways that depend on the scaling a. For $a = 1000$, it resembles the rising note of a siren; for $a = 2500$, it is a banshee howl; and for $a = 5395$ (a choice explained later) it is an unnerving scream.

These sounds can be understood by representing $Z(t)$ as a series of oscillations. To sufficient accuracy this given by the 'main sum' of the Riemann–Siegel expansion [1, 15]—a version of the Dirichlet series incorporating the functional equation for $\zeta(s)$:

$$Z_{\mathrm{RS}}(t) = 2\sum_{n=1}^{\lfloor\sqrt{t/2\pi}\rfloor} \frac{\cos\{\theta(t) - t\log n\}}{\sqrt{n}}. \tag{5.3}$$

Because of the floor (integer part) function $\lfloor\sqrt{t/2\pi}\rfloor$, this is a finite sum in which the number of terms increases slowly with t. The corresponding weak discontinuities are barely audible when $Z_{\mathrm{RS}}(t)$ is rendered as a sound (and in any case could be eliminated by including Riemann–Siegel correction terms [15] or by smoothing the discontinuities [16]).

The oscillations in (5.3) involve $\theta(t)$, whose large-t asymptotic form is

$$\theta(t) \approx \frac{1}{2}t\log\left(\frac{t}{2\pi\mathrm{e}}\right) - \frac{1}{8}\pi. \tag{5.4}$$

Thus the instantaneous frequencies of the oscillations with indices n are

$$\nu_n(t) = \frac{1}{2\pi}\frac{\mathrm{d(phase)}}{\mathrm{d}t} \approx \frac{1}{2\pi}\log\left(\frac{1}{n}\sqrt{\frac{t}{2\pi}}\right). \tag{5.5}$$

These can be regarded as rising tones, quasi-monochromatic because the frequencies scarcely vary over an oscillation period:

$$\frac{\nu_n(t+1/\nu_n(t)) - \nu_n(t)}{\nu_n(t)} \approx \frac{1}{\nu_n(t)^2}\frac{\mathrm{d}\nu_n(t)}{\mathrm{d}t} = \frac{\pi}{t\log^2\left(\frac{1}{n}\sqrt{\frac{t}{2\pi}}\right)} \ll 1 \tag{5.6}$$

for all relevant t and n. The logarithmic spectrum (5.5) contrasts with the exponential spectrum $\nu_n = \nu_1 2^{n/12}$ of the semitones of the musical scale.

In the scaled version $Z_{\mathrm{RS}}(at)$ of the sum (5.3), the highest frequency corresponds to $n = 1$:

$$\nu_{\max}(t) = \frac{a\log\left(\frac{at}{2\pi}\right)}{4\pi}. \tag{5.7}$$

For a sound played for the time interval $0 \leqslant t \leqslant t_{\max}$, a can be determined to correspond to any choice of the highest frequency $\nu_{\max}(t_{\max})$ in this zeta music. For $t_{\max} = 20$ s, and the highest note C8 = 4186 Hz on the piano keyboard (equation (3.2)), this gives $a = 5395$; $a = 2500$ corresponds to $\nu_{\max}(20) = 1787$ Hz, and $a = 1000$ corresponds to $\nu_{\max}(20) = 642$ Hz.

The very different sound of the zeta music $Z(t)$ as compared with the prime music $S(\tau)$ (equation (2.6)) is reflected in the power spectrum. In terms of the angular frequency $\omega = 2\pi\nu$, inverting (5.5) gives

$$n(\omega) = \sqrt{\frac{t}{2\pi}}\exp(-\omega). \tag{5.8}$$

Thus the power spectrum corresponding to (5.3) is

$$P(\omega) = \frac{1}{n(\omega)}\left|\frac{\mathrm{d}n(\omega)}{\mathrm{d}\omega}\right| = \Theta\left(\log\sqrt{\frac{t}{2\pi}} - \omega\right). \tag{5.9}$$

Ignoring the step function Θ, this is a flat spectrum, corresponding to white noise. This was noted in a previous study [17, pp 253–260], emphasizing the very different spectra of $\zeta(\sigma + \mathrm{i}t)$ on and off the critical line $\sigma = 1/2$. However, the cutoff is important because it indicates that the zeta music is band-limited. Over sufficiently long times, the cutoff frequency $\nu_{\max}$ in (5.7) will rise above the audible range. Even then, the logarithmic distribution of frequencies (5.5) sounds different from simulated white noise in which the frequencies ($\sqrt{t/2\pi}$ of them) are uniformly distributed throughout the range $0 < \nu < \nu_{\max}$. And of course the flat band-limited spectrum of zeta is very different from the fractal spectrum of the prime music as discussed in section 4.

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