commentary

Geometric phase memories

Michael Berry

The moment of conception of the geometric phase can be pinpointed precisely, but related ideas had been formulated before, in various guises. Not less varied were the ramifications that became clear once the concept was identified formally.

A little more than 25 years ago, I introduced the geometric phase: when the parameters of a quantum mechanical wavefunction are slowly cycled around a circuit, then the phase of the wavefunction need not return to its original value[1]. In recalling the events surrounding the publication of that paper, I should back up to the early 1970s, to Ian Percival's pioneering insight[2] that classical chaos — then unfamiliar to physicists — must have implications in quantum mechanics, in particular for the spectrum of discrete energy levels of systems whose classical motion would be chaotic, such as atoms in strong magnetic fields or molecules whose atoms interact anharmonically.

My colleague Balazs Gyorffy suggested that such 'irregular spectra' might be described by random-matrix theory, which had been developed to understand the statistics of energy levels in nuclei[3]. A central feature of spectra of random matrices is that energy levels repel. For the universality class of systems with time-reversal symmetry (where spin is unimportant) level repulsion is described quantitatively by the probability distribution of spacings S between nearest-neighbour levels: the probability of finding a given spacing vanishes linearly for near-degeneracies, that is, as S goes to zero. Following insights from von Neumann and Wigner[4], it soon became clear that linear repulsion could be understood as the shadow of true degeneracies (where $S = 0$) in nearby systems in which the system under consideration is imagined to be embedded. These degeneracies require two parameters — one is in general not sufficient to produce a degeneracy — and in terms of these parameters the energy levels are sheets in the form of a double cone. The double cone is also called a diabolo (after a spinning toy of the same shape), so I called the intersections 'diabolical points'.

But how can we know that the two sheets really touch, rather than avoiding each other as energy levels typically do when just one parameter is varied? In 1978 I found the criterion: while encircling a diabolical point in the space of parameters, each of the two wavefunctions, when smoothly continued round its sheet, must change sign[5]. This simple topological result was very satisfying — a property of 2 × 2 matrices that should be in every textbook. But apparently it was in none. Alas, I quickly learned that my 'discovery' was not original: several years earlier, the mathematician Karen Uhlenbeck had written about the sign change[6], and, nearly two decades before that, Christopher Longuet-Higgins[7] and others had found it in a study of energy levels in molecules (the 'others' were physicists in Bristol, but more about this later).

I wanted to see these theoretical constructs — diabolical points, and the associated sign change — in computations for a concrete class of systems. Michael Wilkinson and I chose to explore the spectra of triangular quantum billiards: energy states of particles confined in a triangular domain with hard walls, equivalent to the vibration modes of a triangular drum. Triangles are indeed described by two parameters, namely any two angles. We found several diabolical points[8] and confirmed their existence by calculating the sign change.

The moment of conception

In the spring of 1983, I talked about this work at the Georgia Institute of Technology, emphasizing the importance of time reversal. If this symmetry is broken, the spectra would fall into a different, more general universality class, in which degeneracies would typically require the explorations of three parameters, not two (ref. 4). So, if a weak magnetic field were added to the particle in the triangles, the diabolical points would disappear. At the end of the talk, Ronald Fox (at that time the chairman of the physics department) asked what happens to the sign change when the magnetic field is switched on.

This was the trigger, the moment of conception. My immediate response, "I suppose it's a phase change different from π", was followed by the premature "I'll work it out tonight and tell you tomorrow". In fact it took several weeks to understand the geometric phase properly. The outcome was an unexpected general feature of quantum mechanics: although wavefunctions are single-valued in the space of variables on which they depend, they need not be single-valued under continuation around a circuit of parameters that drive them. And when the driving is slow — 'adiabatic' — the mathematically natural continuation was exactly the one that is enforced by the time-dependent Schrödinger equation. After checking every book on quantum mechanics in our library, and failing to find the geometric phase effect described or even suggested, I decided to write up the work for publication.

John Hannay, who had made useful suggestions while the research was progressing, urged that my proposed title, which included the term 'topological phase factors', was misleading. In the general case, the phase depends continuously on the shape of the circuit in the space of parameters, so the phenomenon I had identified was geometric, not topological. Only in special cases — systems with time-reversal symmetry, or the Aharonov–Bohm effect — is the phase topological.

Anticipations and ramifications

After writing the paper[1], but before submitting it for publication, I enjoyed a long drive through the English countryside with Eric Heller, during which I told him enthusiastically about the phase, and asked if he knew of anything similar. He did. Several years before, Alden Mead and Don Truhlar had identified 'the molecular Aharonov–Bohm effect' in quantum chemistry[9]. Back in Bristol, I anxiously looked at their paper, and found to my dismay that they had indeed anticipated several features of the geometric phase. Fortunately I was able to cite their work in my paper, hoping it retained sufficient novelty and generality to survive the referees' scrutiny.

I sent it to *Proceedings of the Royal Society of London*, my journal of choice for work I was particularly pleased with (at present, I am the editor). The paper was received on

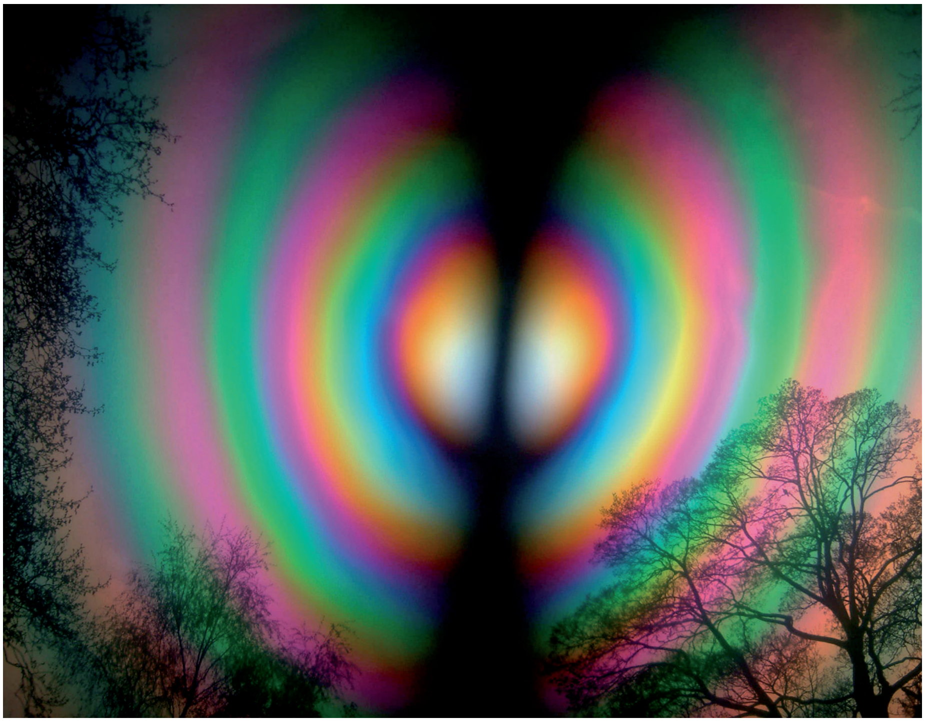

Figure 1 | A polarization bull's-eye singularity photographed in the sky above Bristol University. Between the camera lens and the sky was a 'black-light sandwich', consisting of a sheet of overhead-projector transparency film (which happens to be biaxially anisotropic) sandwiched between two orthogonal polarizing filters[49]. The black stripe is a manifestation of the geometric phase.

13 June 1983. As the journal later informed me, one referee eventually confessed to having lost the manuscript, and it was only in 1984 that it was finally published. Meanwhile, I had met Barry Simon in Australia, and told him about the phase. Overnight, he recognized it as a manifestation of anholonomy as known in fibre bundle theory: the failure of some quantities to return when others, that drive them, are cycled. His paper[10], describing these connections with mathematics and also with earlier work on the quantum Hall effect[11], was published in late 1983. After it appeared in *Physical Review Letters*, and he coined the term 'Berry's phase', the concept, and my subsequent publication, became widely known.

Generalizations and extensions soon followed: from Wilczek and Zee[12] to the non-Abelian case where a collection of degenerate states, rather than a single one, is cycled; from John Hannay[13,14] to an analogous phase in classical mechanics; from Yakir Aharonov and Jeeva Anandan[15] to a formulation that did not require adiabaticity; from John Garrison and Ewan Wright[16] to non-Hermitian evolution.

A particularly far-reaching extension was Hiroshi Kuratsuji and Shini Iida's interpretation[17] of the driving parameters as dynamical variables, whose evolution was influenced by the same geometrical objects as the phase. It became clear[18] that the reaction of the geometric phase on the parameters takes the form of an abstract magnetic field, which I explored with Jonathan Robbins[19]; we called it 'geometric magnetism'. We now know that this is the first of a hierarchy of geometric reaction forces on a slow system coupled to a fast one. Very recently, I have returned to this subject, in a study, with Pragya Shukla, of the infinite series of reaction forces[20]; several features of the separation between fast and slow variables, however, remain mysterious.

I soon learned about other anticipations of the phase than Mead and Truhlar's. While visiting India in 1986, Rajaram Nityananda and Sivaraj Ramaseshan showed me the paper they had written[21] reviving the discovery, by Shivaramakrishnam Pancharatnam[22] 30 years before, of a geometric phase in beams of light whose polarization state is cycled (see Fig. 1). On the long flight home, I made the connection between Pancharatnam's optical picture and my general quantum formalism[23,24], and realized that he had formulated the phase for two-state systems. Pancharatnam was one of the several physicist nephews of Chandrasekhara Venkata Raman; in 1956, when he discovered his polarization phase, he was only 22 years old. Another early version was the discovery by Kenneth Budden and Martin Smith[25] of 'additional phase memory' in radiowave propagation. These anticipations illustrate that in retrospect what we call 'discovery' sometimes looks more like emergence into the air from subterranean intellectual currents[26].

The phase emerged from my earlier interest in semiclassical physics, but influenced my subsequent intellectual trajectory in a very different way. The adiabatic formulation, in which the state is driven slowly, raised the question of corrections to the phase, of higher order in slowness. A calculation, in 1987, of the infinite series of these higher orders, involved the recognition that the series must diverge if there are to be any non-adiabatic transitions between the cycled state and other states[27]. As transitions between states represent the way in which quantum mechanics describes events (in contrast to eigenstates, which describe things), there is a sense in which the divergence of the series is necessary in order for anything to happen. This was the beginning of a series of developments[28,29] continuing into the 1990s and building on earlier seminal insights by Robert Dingle[30] (who had been my doctoral supervisor in the 1960s), in which increasingly sophisticated ways of summing divergent series were devised[31,32], and the inevitability of divergence in series arising throughout theoretical physics[33,34] became clear.

An unavoidable discovery?

It took several years to appreciate that although my understanding of the geometric phase involved a series of accidents — the quantum physics of classical chaos, Ronald Fox's question, the drive with Eric Heller, and others — there was a certain inevitability about it emerging from the physics department of Bristol University. The reason is that physics associated with circuits was embedded in the culture of the department (see also ref. 35). Its inspiration was the deeply geometric personality of Charles Frank, who worked in Bristol from 1946 until his death in 1998. In 1951, he gave the definitive understanding[36] of dislocations in solids in terms of a 'Burgers circuit' of a defect in the real crystal, whose image in a fictitious ideal crystal failed to close. In 1958 he applied similar ideas[37] to classify defects in liquid crystals. In the following year, Longuet-Higgins and a group including Maurice Pryce, then the head of the physics department in Bristol, understood the sign change associated with molecular electronic degeneracies[7]. At the same time, Yakir Aharonov and David Bohm discovered their celebrated eponymous effect[38], interpretable as the phase change that a quantum electron acquires during a circuit of a line of magnetic flux; soon afterwards, Robert Chambers[39] gave the first experimental demonstration. In 1974, John Nye and I identified phase singularities[40] (also termed wavefront dislocations, nodal lines or wave vortices) as ubiquitous features of waves, classical or quantum. And in the 1980s, Nye gave a similar characterization of polarization singularities[41,42], associated with singularities in vector waves. With this perspective, finding the geometric phase at Bristol appears unsurprising.

After 1983, my interests shifted. I did not follow the many applications of geometric phases in different areas of physics, and so cannot review them. But others have[43–48]. ❒

Michael Berry is in the H. H. Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TK, UK.
e-mail: Tracie.Anderson@bristol.ac.uk

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