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Optical lattices with PT symmetry are not transparent

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Abstract
In diffraction of a plane wave by a non-Hermitian PT symmetric optical lattice, the sum of the Bragg beam intensities need not be conserved, even though the gain and loss are equally distributed: the evolution is not unitary. Instead, different sums are conserved, in which the intensities are weighted with real numbers (positive or negative); several such sum rules are derived. Two-beam diffraction from a refractive index of the form constant $-a\cos x + ib\sin x$ is studied in detail; the sum rule depends on the balance between the (real) Hermitian parameter a and the (real) anti-Hermitian parameter b.

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1. Introduction

Following the demonstration that simple PT-symmetric non-Hermitian potentials can have real spectra, it has been shown that such Hamiltonians can be incorporated consistently into a modified version of quantum mechanics in which evolution is unitary, involving a new kind of scalar product (for reviews, see [1, 2]. Now there are proposals for creating non-Hermitian PT-symmetric systems in the laboratory, for example in optical lattices [3]. In one implementation, this would involve a volume grating in the form of a slab (figure 1(*a*)) with periodic transverse variation of refractive index $n(x)$, which is PT-symmetric because Re $n(x)$ is even and Im $n(x)$ is odd. We will call this lattice a 'PT crystal'. Im $n(x) > 0$ corresponds to loss (absorption) and Im $n(x) < 0$ corresponds to gain. A plane wave of light is incident near the z direction, and emerges in a series of Bragg-diffracted beams. The language is that of optics, but the arguments apply to PT crystals interacting with any wave, for example neutrons or atoms satisfying the Schrödinger equation.

The natural question arises: in a PT crystal, is the evolution unitary in the sense that the sum of intensities of the emerging Bragg beams is equal to the incident intensity, so the lattice can be called transparent? A suggestion of a positive answer—that evolution might be unitary—comes from the fact that in a PT crystal gain and loss are equally distributed, because Im $n(x)$ is odd. Nevertheless, the answer is no: PT crystals are not transparent because the waves evolving inside them do not explore the gain and loss regions democratically.

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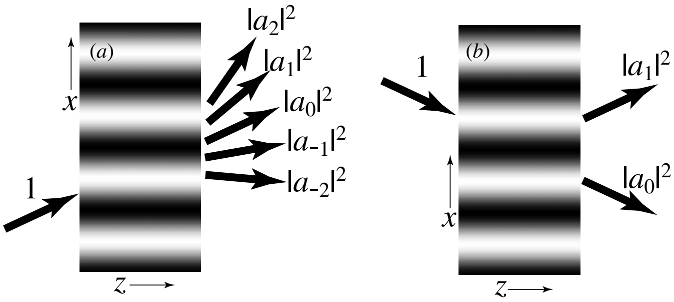

Figure 1. (*a*) Geometry of Bragg diffraction from an optical lattice; (*b*) two-beam case.

To demonstrate nontransparency, I show for the simplest model and for several types of PT crystals that there exists a conserved sum over the emerging beams, but the intensities are unequally weighted, in contrast to the equal weighting in the sum rule corresponding to unitary evolution. Section 2 sets up the formalism, section 3 derives the sum rules, and section 4 explores the simplest nontrivial case of two-beam diffraction (figure 1(*b*)).

Although the idea of creating PT crystals, with loss balanced by gain, is new, the analysis of some previous experiments with space-varying purely absorbing lattices is essentially the same, since the mean loss simply represents an overall exponential decay of the wave. An example of such 'effective PT symmetry' has been demonstrated in atom optics [4, 5]. Related non-Hermitian phenomena in absorbing crystals are the critical voltage effect in electron microscopy [6, 7], and the Borrmann effect in x-ray diffraction [7].

2. Formulation

Waves in the lattice are assumed to satisfy the scalar Helmholtz equation

$$\partial_z^2\Psi + \partial_x^2\Psi + k^2 n(x)^2\Psi = 0. \tag{1}$$

The following scalings are convenient, involving the incident wavenumber k, the grating period l and the average refractive index n_0:

$$\begin{aligned} &\Psi(x,z) \equiv \exp(\mathrm{i}kn_0 z)\psi(\xi,\zeta), \qquad \xi \equiv \frac{x}{l}, \\ &\zeta \equiv \frac{z}{2kl^2 n_0}, \qquad \mu(\xi) \equiv k^2 l^2\left(n_0^2 - n(x)^2\right) = \mu(\xi + 2\pi). \end{aligned} \tag{2}$$

The refractive index of the PT crystal, now represented by $\mu(\xi)$, is the sum of a Hermitian and an anti-Hermitian part, with the following properties:

$$\begin{aligned} &\mu(\xi) = \mu_{\mathrm{h}}(\xi) + \mu_{\mathrm{a}}(\xi) \\ &\mu_{\mathrm{h}}(\xi) \text{ (Hermitian) real even} \\ &\mu_{\mathrm{a}}(\xi) \text{ (anti-Hermitian) imaginary odd.} \end{aligned} \tag{3}$$

Im $\mu_{\mathrm{a}}(\xi) < 0$ corresponds to loss, and Im $\mu_{\mathrm{a}}(\xi) > 0$ to gain. PT symmetry implies that the Fourier coefficients μ_n, defined by

$$\mu(\xi) = \sum_{n=-\infty}^{\infty} \mu_n \exp(\mathrm{i}n\xi), \tag{4}$$

are real, with the Hermitian part even in n and the anti-Hermitian part odd in n:

$$\mu_n = \text{real} = \mu_{\text{h}n} + \mu_{\text{a}n}, \qquad \mu_{\text{h}n} = \mu_{\text{h},-n}, \qquad \mu_{\text{a}n} = -\mu_{\text{a},-n}. \tag{5}$$

The incident plane wave is inclined at an angle θ_0 to the z direction, represented as a multiple α_0 of the Bragg angle by

$$\sin\theta_0 = \alpha_0/kl. \tag{6}$$

For simplicity, it is assumed that $\theta_0 \ll 1$ and $\mu(\xi)$ is small enough to justify the paraxial approximation, which with the scalings (2) takes the form

$$\mathrm{i}\partial_\zeta\psi = -\partial_\xi^2\psi + \mu(\xi)\psi. \tag{7}$$

Because of the periodicity, the wave in the lattice can be written

$$\psi(\xi,\zeta) = \sum_{n=-\infty}^{\infty} a_n(\zeta)\exp\{\mathrm{i}(n+\alpha_0)\xi\}, \tag{8}$$

in which $a_n(\zeta)$ is the amplitude of the nth Bragg beam, with $n = 0$ denoting the undeflected beam, and $|a_n(\zeta)|^2$ is the nth Bragg intensity. Substitution into (7) gives the coupled differential equations

$$\mathrm{i}\partial_\zeta a_n(\zeta) = (n+\alpha_0)^2 a_n(\zeta) + \sum_{m=-\infty}^{\infty}\mu_{n-m}a_m(\zeta), \qquad a_n(0) = \delta_{n,0}. \tag{9}$$

Later it will sometimes be convenient to write the evolution equation in Dirac notation as

$$\mathrm{i}\partial_\zeta|a(\zeta)\rangle = \mathbf{H}|a(\zeta)\rangle, \tag{10}$$

in which $|a(\zeta)\rangle$ is the vector with components $a_n(\zeta)$, and the Hamiltonian is

$$\mathbf{H} = \{H_{mn} = (n+\alpha_0)^2\delta_{mn} + \mu_{n-m}\}. \tag{11}$$

A non-Hermitian PT crystal, that is one with loss and gain, for which $\mu_\text{a}(\xi) \neq 0$ in (3), is here represented by a Hamiltonian matrix that is real but not symmetric. Thus the secular equation is real, and the eigenvalues, which determine the propagation, have the property, shared by all PT-symmetric Hamiltonians [8] of being either real or forming complex-conjugate pairs. The latter situation is termed 'broken PT symmetry' [2], and the borderline situation of the birth of two complex eigenvalues is a non-Hermitian degeneracy [9], often referred to as an 'exceptional point'.

3. Intensity sum rules

The generalized sum rules we seek take the form

$$S \equiv \sum_{n=-\infty}^{\infty} S_n|a_n(\zeta)|^2 = 1, \tag{12}$$

in which the S_n are real numbers that may be positive or negative. The requirement that S is conserved (independent of ζ) is equivalent to

$$\partial_\zeta S = 0. \tag{13}$$

From (9), the rate of change of the individual intensities is

$$\partial_\zeta|a_n|^2 = 2\,\text{Im}\sum_{m=-\infty}^{\infty} a_n^* a_m\mu_{n-m}. \tag{14}$$

Thus, for any lattice, not necessarily a PT crystal,

$$\begin{aligned}\partial_\zeta S &= 2\,\mathrm{Im}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} S_n a_n^* a_m \mu_{n-m}\\ &= 2\,\mathrm{Im}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} S_m a_m^* a_n \mu_{m-n}\\ &= -2\,\mathrm{Im}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} S_m a_n^* a_m \mu_{m-n}^*\\ &= \mathrm{Im}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \left(S_n\mu_{n-m} - S_m\mu_{m-n}^*\right) a_n^* a_m. \end{aligned} \tag{15}$$

In the simple case of a general Hermitian lattice, which is transparent (neither loss nor gain) and which need not have PT symmetry,

$$\mu_{-n} = \mu_n^*, \tag{16}$$

so

$$S_n\mu_{n-m} - S_m\mu_{m-n}^* = \mu_{n-m}(S_n - S_m), \tag{17}$$

which is satisfied by $S_n = S_m = 1$, giving the familiar conserved total intensity

$$\sum_{n=-\infty}^{\infty} |a_n(\zeta)|^2 = \text{constant} = \sum_{n=-\infty}^{\infty} |a_n(0)|^2 = 1. \tag{18}$$

In the first nontrivial case to be considered, $\mu(\xi)$ is doubly restricted. First, there is no Hermitian part, that is $\mu_{\mathrm{h}n} = 0$ ('pure non-Hermitian PT crystal'), so

$$S_n\mu_{n-m} - S_m\mu_{m-n}^* = (S_n + S_m)\mu_{\mathrm{a},n-m}. \tag{19}$$

Second, the anti-Hermitian part has no even Fourier components. Thus

$$\mu(\xi) = 2\mathrm{i}\sum_{n=1}^{\infty} \mu_{\mathrm{a},2n+1}\sin\{(2n+1)\xi\}, \tag{20}$$

representing crystals odd with respect to $\xi = \pi$ as well as $\xi = 0$ (or, alternatively stated, even with respect to $\xi = \pi/2$). Then $n-m$ is odd in (19), which can therefore be satisfied by

$$S_n = (-1)^n, \tag{21}$$

leading to (13) and the alternating-sign sum rule

$$\sum_{n=-\infty}^{\infty} (-1)^n |a_n(\zeta)|^2 = \text{constant} = \sum_{n=-\infty}^{\infty} (-1)^n |a_n(0)|^2 = 1. \tag{22}$$

In such PT crystals, the total intensity satisfies the inequality

$$\sum_{n=-\infty}^{\infty} |a_n(\zeta)|^2 = 1 + \sum_{n=-\infty}^{\infty} |a_{2n+1}(\zeta)|^2 \geqslant 1, \tag{23}$$

that is, gain always dominates loss, demonstrating that these crystals are not transparent.

For PT crystals in which the only nonvanishing refractive-index coefficients are μ_1 and μ_{-1}, the transition between the Hermitian and alternating-sign sum rules can be exhibited explicitly. For such lattices,

$$\mu(\xi) = \mu_1\exp(\mathrm{i}\xi) + \mu_{-1}\exp(-\mathrm{i}\xi) = 2\,\mu_{\mathrm{h}1}\cos\xi + 2\mathrm{i}\,\mu_{\mathrm{a}1}\sin\xi. \tag{24}$$

In this case, the requirement that the last member in (15) vanishes is

$$S_n\mu_1 - S_{n-1}\mu_{-1} = 0. \tag{25}$$

The solution is

$$S_n = \left(\frac{\mu_{\mathrm{h}1} - \mu_{\mathrm{a}1}}{\mu_{\mathrm{h}1} + \mu_{\mathrm{a}1}}\right)^n, \tag{26}$$

yielding the sum rule

$$\sum_{n=-\infty}^{\infty} \left(\frac{\mu_{\mathrm{h}1} - \mu_{\mathrm{a}1}}{\mu_{\mathrm{h}1} + \mu_{\mathrm{a}1}}\right)^n |a_n(\zeta)|^2 = 1. \tag{27}$$

This reproduces (18) in the Hermitian case $\mu_{\mathrm{a}1} = 0$, and the pure non-Hermitian PT case (22) when $\mu_{\mathrm{h}1} = 0$.

It is natural to ask if a sum rule exists for a general PT crystal, in which all coefficients $\mu_{\mathrm{h}n}$ and $\mu_{\mathrm{a}n}$ can be nonzero. I do not know the answer, but offer the following formulation of the problem, in terms of an operator defined by the diagonal matrix of the desired sum rule coefficients:

$$\mathbf{S} \equiv \{S_n\delta_{nm}\}. \tag{28}$$

The sum in (12) can be written

$$S = \langle a(\zeta)|\mathbf{S}|a(\zeta)\rangle = \langle a(0)|\exp(\mathrm{i}\zeta\mathbf{H}^\dagger)\mathbf{S}\exp(-\mathrm{i}\zeta\mathbf{H})|a(0)\rangle, \tag{29}$$

whose conservation requires

$$\exp(\mathrm{i}\zeta\mathbf{H}^\dagger)\mathbf{S}\exp(-\mathrm{i}\zeta\mathbf{H}) = S, \qquad \text{i.e. } \mathbf{SHS}^{-1} = \mathbf{H}^\dagger. \tag{30}$$

It is an interesting exercise (not given here) to verify this for the three cases already considered, but the general case remains open.

For a general **S**, not necessarily diagonal as in (28), (30) is the condition for **H** to be pseudo-Hermitian [10]. However, if **S** is not diagonal the conserved quantity S in (29) contains cross-terms of the form $a_m^* a_n$, and so is not an intensity sum rule in the sense of involving only the quantities $|a_n|^2$; we will see an example later.

4. Two-beam example

Here we consider the PT crystal (24), for which if $\mu_{\mathrm{a}1} > 0$ the region $0 < \xi < \pi$ corresponds to gain and $\pi < \xi < 2\pi$ corresponds to loss. We also assume $|\mu_{\mathrm{h}1}| \ll 1$ and $|\mu_{\mathrm{a}1}| \ll 1$, and light incident close to the Bragg reflection angle (figure 1(*b*)), that is

$$\alpha_0 = -\tfrac{1}{2} + \delta, \qquad |\delta| \ll 1. \tag{31}$$

Then all coefficients in the wave (8) are negligible except a_0 and a_1, and the recurrence relation (9) can be expressed in terms of a 2×2 truncation of the matrix Hamiltonian (11):

$$\begin{aligned}\mathbf{H} &= \left(\delta^2 + \frac{1}{4}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}-\delta & \mu_{-1}\\ \mu_1 & \delta\end{pmatrix}\\ &= \left(\delta^2 + \frac{1}{4}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}-\delta & \mu_{h1} - \mu_{a1}\\ \mu_{h1} + \mu_{a1} & \delta\end{pmatrix}.\end{aligned} \tag{32}$$

The eigenvalues,

$$\begin{aligned}\lambda_\pm &= \delta^2 + \tfrac{1}{4} \pm \sqrt{\delta^2 + \mu_1\mu_{-1}}\\ &= \delta^2 + \tfrac{1}{4} \pm \sqrt{\delta^2 + \mu_{\mathrm{h}1}^2 - \mu_{\mathrm{a}1}^2} \equiv \bar{\lambda} \pm \tfrac{1}{2}\Delta\lambda,\end{aligned} \tag{33}$$

are real if $\mu_{a1}^2 < \delta^2 + \mu_{h1}^2$ (unbroken PT symmetry) and complex conjugates if $\mu_{a1}^2 > \delta^2 + \mu_{h1}^2$ (broken PT symmetry), with a non-Hermitian degeneracy when $\mu_{a1}^2 = \delta^2 + \mu_{h1}^2$, corresponding to the eigenvalue $\lambda_\pm = \bar{\lambda} = \delta^2 + \frac{1}{4}$. For a given PT crystal, that is μ_{a1} and μ_{h1} fixed, degeneracy corresponds to incident light directions

$$\delta = \pm\sqrt{\mu_{a1}^2 - \mu_{h1}^2}, \tag{34}$$

provided $\mu_{a1}^2 > \mu_{h1}^2$, that is, the anti-Hermitian part of the refractive index dominates the Hermitian part. Between the directions (34), spanning the Bragg angle $\delta = 0$, PT symmetry is broken; outside this range, the symmetry is unbroken. In the opposite situation, of a Hermitian-dominated lattice, that is $\mu_{a1}^2 < \mu_{h1}^2$, there are no degeneracies and the PT symmetry is always unbroken.

Evolution of the wave in the lattice is determined by the 2×2 matrix exponential

$$\begin{aligned}\exp(-\mathrm{i}\zeta\mathbf{H}) = \exp\{-\mathrm{i}\zeta\bar{\lambda}\}\Bigg[&\cos\left(\frac{1}{2}\zeta\Delta\lambda\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &- \frac{2\mathrm{i}}{\Delta\lambda}\sin\left(\frac{1}{2}\zeta\Delta\lambda\right)\begin{pmatrix}-\delta & \mu_{h1}-\mu_{a1}\\ \mu_{h1}+\mu_{a1} & \delta\end{pmatrix}\Bigg].\end{aligned} \tag{35}$$

Thus the amplitudes are

$$\begin{aligned}a_0(\zeta) &= \left[\cos\left(\frac{1}{2}\zeta\Delta\lambda\right) + \frac{2\mathrm{i}\delta}{\Delta\lambda}\sin\left(\frac{1}{2}\zeta\Delta\lambda\right)\right]\exp\{-\mathrm{i}\zeta\bar{\lambda}\}\\ a_1(\zeta) &= -\frac{2\mathrm{i}(\mu_{h1}+\mu_{a1})}{\Delta\lambda}\sin\left(\frac{1}{2}\zeta\Delta\lambda\right)\exp\{-\mathrm{i}\zeta\bar{\lambda}\}.\end{aligned} \tag{36}$$

For unbroken PT symmetry, $\Delta\lambda$ is real and the amplitudes oscillate. For broken PT symmetry, $\Delta\lambda$ is imaginary and the sin function is hyperbolic, giving exponential variation of the amplitudes. In both cases, they satisfy the sum rule (27), which for this two-beam case is

$$|a_0(\zeta)|^2 + \left(\frac{\mu_{h1}-\mu_{a1}}{\mu_{h1}+\mu_{a1}}\right)|a_1(\zeta)|^2 = 1. \tag{37}$$

The total intensity is

$$|a_0(\zeta)|^2 + |a_1(\zeta)|^2 = 1 + \frac{2\sin^2\left(\frac{1}{2}\zeta\Delta\lambda\right)\mu_{a1}(\mu_{h1}+\mu_{a1})}{\delta^2 + \mu_{h1}^2 - \mu_{a1}^2}. \tag{38}$$

If $\mu_{a1}^2 < \mu_{h1}^2$, i.e. the symmetry is always unbroken, and μ_a and μ_h have opposite signs, the total intensity is always less than unity: loss dominates gain. In all other cases, which may correspond to broken or unbroken symmetry depending on the value of δ, the total intensity exceeds unity: gain dominates loss. (Note that the factor $\sin^2\left(\frac{1}{2}\zeta\Delta\lambda\right)/\left(\delta^2 + \mu_{h1}^2 - \mu_{a1}^2\right)$ is never negative.)

At the degeneracy, that is with incident direction (34), (36) reduces to

$$\begin{aligned}a_0(\zeta) &= \left[1 + \mathrm{i}\zeta\sqrt{\mu_{a1}^2 - \mu_{h1}^2}\right]\exp\{-\mathrm{i}\zeta\bar{\lambda}\}\\ a_1(\zeta) &= -\mathrm{i}\zeta(\mu_{h1}+\mu_{a1})\exp\{-\mathrm{i}\zeta\bar{\lambda}\}.\end{aligned} \tag{39}$$

As ζ increases, the state rotates to become parallel to the single degenerate eigenvector of (32), namely

$$\begin{pmatrix}a_0(\zeta)\\ a_1(\zeta)\end{pmatrix} \propto \begin{pmatrix}\sqrt{\mu_{a1}-\mu_{h1}}\\ -\sqrt{\mu_{a1}+\mu_{h1}}\end{pmatrix}. \tag{40}$$

As has been noted before, for example in crystal optics [11–13], this linear growth and rotation are characteristic of evolution generated by operators with a non-Hermitian degeneracy. The intensities are

$$|a_0|^2 = 1 + \zeta^2\left(\mu_{a1}^2 - \mu_{h1}^2\right), \qquad |a_1|^2 = \zeta^2(\mu_{h1} + \mu_{a1})^2, \tag{41}$$

so in this marginal case, as with broken PT symmetry, gain dominates loss: the total intensity (which satisfies the sum rule (37)) always exceeds unity because $\mu_{a1}^2 > \mu_{h1}^2$.

The wave intensity at the degeneracy is, from (8) and (39),

$$|\psi(\xi,\zeta)|^2 = 1 + 2\zeta^2(\mu_{a1} + \mu_{h1})^2\left(1 - \sqrt{\frac{\mu_{a1} - \mu_{h1}}{\mu_{a1} + \mu_{h1}}}\cos\xi\right) + 2\zeta(\mu_{a1} + \mu_{h1})\sin\xi. \tag{42}$$

The last term breaks the symmetry between the gain and loss regions, as claimed in the Introduction, and is larger in the gain region (as is physically obvious). It follows from (36) that the same result holds away from the degeneracy when PT symmetry is broken.

Finally, an example of a nondiagonal matrix **S**, satisfying (30) with the two-beam Hamiltonian **H** (32), is just the nontrivial symmetric part of **H**:

$$\mathbf{S} = \begin{pmatrix} -\delta & \mu_{h1} \\ \mu_{h1} & \delta \end{pmatrix}. \tag{43}$$

The corresponding conserved quantity (29), which is not an intensity sum rule because it contains cross terms, is

$$|a_0(\zeta)|^2 - |a_1(\zeta)|^2 - \frac{2\mu_{h1}}{\delta}\,\mathrm{Re}\left(a_0^*(\zeta)a_1(\zeta)\right) = 1. \tag{44}$$

This is easily confirmed from (36), for broken as well as unbroken PT symmetry.

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