Nonlinearity 4 (1991) 1–14. Printed in the UK

The parametric number variance

J Goldberg†, U Smilansky†, M V Berry‡, W Schweizer§, G Wunner§ and G Zeller§

† Department of Physics, Weizmann Institute, Rehovot 76100 Israel
‡ H H Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol BS8 1TL, UK
§ Lehrstuhl für Theoretische Astrophysik, Universität Tübingen, D-7400 Tübingen, FRG

Received 21 January 1990
Accepted by J D Gibbon

Abstract. We define a new statistic, the parametric number variance, which measures the correlation of fluctuations in energy levels as a parameter external to the system is varied. A semiclassical formula is obtained and regimes of universal and system dependent behaviour are predicted. Numerical calculations of the PNV in two model systems are found to be in good agreement with the semiclassical theory.

PACS numbers: 0365, 0530, 0545

1. Introduction

A number of statistical measures of the quantum spectra of Hamiltonian systems reflect regularity or chaos in the corresponding classical system. Much numerical [1] and theoretical [2, 3] evidence exists that the levels of a classically integrable system follow a Poisson distribution, while those of a chaotic system with (without) time reversal symmetry are distributed according to the Gaussian orthogonal (unitary) ensemble of random matrices [4].

The most commonly computed statistics are the level spacing ρ, rigidity Δ_3, the number statistic n and its variance Σ^2, asymmetry γ_1 and excess γ_2. Pandey [5] has shown that the Gaussian ensembles are stationary, implying that the various statistics do not depend on which interval of the spectrum is measured. In other words, a statistic calculated by a spectral average on an ensemble member yields the same result as an ensemble average. Indeed, it is customary when comparing real or model systems against random matrix theory predictions to assume stationarity and average over several disjoint energy intervals. Moreover, an 'ensemble average' can be obtained by treating different systems (say nuclei) as effectively independent members of the appropriate ensemble [6].

An important property of the ensembles is that they are composed of independent members. There is no natural measure by which two members can be characterized as being nearby or far apart. In contrast, physical systems often depend upon external (non-dynamic) parameters. A given energy level will depend smoothly on the parameter value, so that the spectra corresponding to nearby values are highly correlated and cannot be considered independent. A plot of energy levels against parameter will

0951-7715/91/010001+14$03.50

show multiple crossings (isolated degeneracies) if the system is regular, or, generically, avoided crossings (near degeneracies) if the system is chaotic [1]. Over a large enough interval in parameter space each of the energy levels of a chaotic system will have many avoided crossings, and the spectra corresponding to end points of the interval will be essentially independent. An effective 'ensemble average' can be obtained from a set of such widely spaced spectra [7].

The parametric number variance is a way to quantify correlations in the fluctuations of neighbouring levels of a chaotic system. It provides a measure of the correlation between nearby spectra and reflects the independence of distant spectra—as well as the minimum distance at which two spectra can be considered uncorrelated. This differs from, but is complementary to, the calculation by Wilkinson [8] of the distribution of avoided crossings (considered pairwise) as a function of a time-dependent parameter.

2. Definition and semiclassical theory

We start with a physical system whose Hamiltonian H is a function of a parameter λ external to the system, and whose jth energy level is denoted $E_j(\lambda)$. The spectral staircase $N(E;\lambda)$ is defined as

$$N(E;\lambda) = \sum_{j=1}^{\infty} \Theta(E - E_j(\lambda)) \tag{1}$$

or in words, the number of states whose energy is less than a given energy E. Here Θ is the unit step function.

Semiclassically, $N(E)$ can be written as the sum of an average and fluctuations:

$$N(E) = \langle N(E) \rangle + N_{\mathrm{osc}}(E) \tag{2}$$

with

$$\langle N(E;\lambda) \rangle = \frac{1}{h^f} \int\int \mathrm{d}q\mathrm{d}p\Theta(E - H(\lambda)) \tag{3}$$

and f the number of degrees of freedom.

As we are interested only in the fluctuations of the spectrum about its average, we normalize the spectrum by defining an 'unfolded spectrum' x_j whose average density is unity, i.e. $x_j \approx j$. The unfolded spectrum is defined as

$$x_j(\lambda) = \langle N(E_j(\lambda);\lambda) \rangle. \tag{4}$$

We define a staircase in x, $N(x;\lambda)$ in complete analogy with (1).

The *parametric number variance* $V(\Lambda)$ is then

$$V(\Lambda) \equiv \langle [N(x;\lambda + \tfrac{1}{2}\Lambda) - N(x;\lambda - \tfrac{1}{2}\Lambda)]^2 \rangle \tag{5}$$

where the $\langle\ \rangle$ means that V is an average over several intervals of length Λ, all at fixed x.

Pictorially, the number variance can be understood as follows: the x versus λ spectrum is composed of roughly horizontal lines, each oscillating about a particular

constant value, as in figures 1 and 3. For any value of x we draw a horizontal line at height x on the spectrum. Then, starting at some arbitrary value λ_0, we count the net number of crossings of the horizontal line by any and all of the x_j from λ_0 to $(\lambda_0 + \Lambda)$. A positive slope crossing gets a value $+1$ and a negative slope crossing -1, with all crossings by each x_j counted. It is here that the correlation in fluctuations of neighbouring levels is felt. Only if a group of levels all move together will the net number of crossings in an interval be two or more.

The net number of crossings is equivalent to the number of levels below x at $(\lambda_0+\Lambda)$ minus the number of levels below x at λ_0. The square of this difference averaged over several intervals of length Λ, but all at the same fixed value x, is the number variance.

The remainder of this section is devoted to calculating $V(\Lambda)$ for chaotic systems. For later convenience, we use a notation that also accommodates regular systems.

It is clear from (5) that $V(\Lambda \to 0) \to 0$. The large Λ behaviour can be deduced by writing (5) as

$$V(\Lambda) = \langle [N(x;\lambda + \tfrac{1}{2}\Lambda) - \langle N(x)\rangle]^2\rangle + \langle [N(x;\lambda - \tfrac{1}{2}\Lambda) - \langle N(x)\rangle]^2\rangle - 2\langle [N(x;\lambda + \tfrac{1}{2}\Lambda) - \langle N(x)\rangle][N(x;\lambda - \tfrac{1}{2}\Lambda) - \langle N(x)\rangle]\rangle. \tag{6}$$

$\langle N(x;\lambda)\rangle = x$ by construction, so each of the first two terms of (6) is simply the number variance, $\Sigma^2(x)$, of the levels in the range 0 to x. The third term vanishes in the limit $\Lambda \to \infty$ as there is no correlation between its two factors. Thus, we expect $V(\Lambda)$ to saturate at $2\Sigma^2(x)$ as Λ becomes large. In particular, for large x, this will be twice the system dependent saturation value $\Sigma^2(\infty)$.

The behaviour of V for intermediate Λ can be calculated semiclassically. Equation (4) defines E implicitly as a function of x and λ. The semiclassical expression for $N(x)$ is given by the Gutzwiller sum over periodic orbits [9] as

$$N(x) = \bar{N}_{\mathrm{osc}}(E) = \frac{-\mathrm{i}}{\hbar^{\mu}} \sum_{j=1}^{\infty} \frac{A_j}{T_j} \mathrm{e}^{\mathrm{i}S_j/\hbar} \tag{7}$$

where the sum is over classical periodic orbits of energy E, T is the orbit period, S is the orbit action and A is a real amplitude factor. μ is $\frac{1}{2}(f-1)$ for regular systems and zero for chaotic systems.

Each of A, T and S is a function of λ, but we may neglect the Λ dependence of A and T compared to $\mathrm{e}^{\mathrm{i}S/\hbar}$ which fluctuates strongly in the limit $\hbar \to 0$. The semiclassical expression for V is thus

$$V(\Lambda) = \frac{1}{\hbar^{2\mu}} \left\langle \sum_{k,j} \frac{A_k A_j}{T_k T_j} \left(\mathrm{e}^{\mathrm{i}S_k(\lambda+\Lambda/2)/\hbar} - \mathrm{e}^{\mathrm{i}S_k(\lambda-\Lambda/2)/\hbar}\right)\left(\mathrm{e}^{-\mathrm{i}S_j(\lambda+\Lambda/2)/\hbar} - \mathrm{e}^{-\mathrm{i}S_j(\lambda-\Lambda/2)/\hbar}\right) \right\rangle. \tag{8}$$

Defining $Q = \partial S/\partial\lambda$ and expanding the phases of (8) to first order, the first factor in parentheses is

$$2\mathrm{i}\mathrm{e}^{\mathrm{i}S_k(\lambda)/\hbar} \sin\left(\frac{\Lambda}{2\hbar} Q_k\right)$$

so that the entire expression becomes

$$V(\Lambda) = \frac{2}{\hbar^{2\mu}} \left\langle \sum_{k,j} \frac{A_k A_j}{T_k T_j} \exp\left(\mathrm{i}\frac{S_k(\lambda) - S_j(\lambda)}{\hbar}\right) \left[\cos\left(\frac{\Lambda(Q_k - Q_j)}{2\hbar}\right) - \cos\left(\frac{\Lambda(Q_k + Q_j)}{2\hbar}\right)\right] \right\rangle. \tag{9}$$

The sum in (9) is over both positive and negative retracings of each primitive periodic orbit, so we rewrite it as

$$V(\Lambda) = \frac{4}{\hbar^{2\mu}}\left\langle \sum_{k,j}^{+} \frac{A_k A_j}{T_k T_j} \cos\left(\frac{S_k(\lambda) - S_j(\lambda)}{\hbar}\right)\left[\cos\left(\frac{\Lambda(Q_k - Q_j)}{2\hbar}\right) - \cos\left(\frac{\Lambda(Q_k + Q_j)}{2\hbar}\right)\right]\right\rangle \tag{10}$$

with the + sign indicating summation over positive traversals only. We have dropped the term $\cos((S_k + S_j)/\hbar)$ from (10). Its contribution vanishes after averaging since all the actions are positive.

Equation (10) is the basic semiclassical formula for V. In order to evaluate it, we will average the Q expression independently of the action (S) expression. This is reasonable, at least for chaotic systems, where we expect the actions to depend delicately on λ so that S and its derivative Q are uncorrelated.

Turning first to the action expression, we make use of an argument to be found in [3]. The form of (10) suggests that its double sum might be diagonalized by a stationary phase argument—that $S_k(\lambda) - S_j(\lambda) \gg \hbar$ for $k \neq j$. This will be true for orbits whose period T is of an intermediate length $T_1 < T < T_2$. T_1 is the time scale for which the set of all orbits of any given $T' > T_1$ fills phase space uniformly. $T_2 = h\langle d\rangle$ is the time beyond which there are exponentially many orbit pairs whose actions differ by less than $\hbar$, a proliferation characteristic of chaotic systems. Nevertheless, beyond T_2 the restriction to pairs whose actions differ by less than $\hbar$ also restricts the sum to pairs of equal periods [3, 10].

Collecting the action expression of (10) into the function $\Phi(T)$ defined [3] by

$$\Phi(T) \equiv \left\langle \sum_{i,j}^{+} A_i A_j \cos[(S_i - S_j)/\hbar]\delta[T - \tfrac{1}{2}(T_i + T_j)] \right\rangle \tag{11}$$

we have that for $T < T_2$ we may use the diagonal form

$$\Phi_{\mathrm{D}}(T) = \left\langle \sum_{j}^{+} A_j^2 \delta(T - T_j) \right\rangle \tag{12}$$

in (10). As the collection of orbits fills phase space, we may use the asymptotic expressions [10]

$$\begin{aligned} \Phi_{\mathrm{D}}(T) &\to \frac{\langle d\rangle \hbar^f}{2\pi} && \text{integrable} \\ \Phi_{\mathrm{D}}(T) &\to \frac{T}{2\pi^2} && \text{chaotic, time reversal} \\ \Phi_{\mathrm{D}}(T) &\to \frac{T}{4\pi^2} && \text{chaotic, no time reversal} \end{aligned} \tag{13}$$

for $T_1 < T < T_2$ and where $\langle d\rangle$ is the average density of states.

For $T > T_2$, we use the semiclassical sum rule of [3]

$$\Phi(T) \to \frac{\langle d\rangle \hbar^{2\mu+1}}{2\pi} \qquad T \gg T_2 \tag{14}$$

which is the same as the long time limit of Φ_D in the regular case. This allows us to use only the diagonal part of (10) when evaluating it for regular systems. In particular, this means that $Q_k = Q_j$ and the term in square brackets of (10) becomes $2\sin^2(\Lambda Q/2\hbar)$.

In the chaotic case we may use the diagonal sum when evaluating the Q expression for $T < T_2$. We make this simplifying assumption for the long orbits as well—an approximation that is consistent with the relation $\Lambda Q \ll S$ implicit in the expansion (9).

We now evaluate the diagonalized Q expression of (10), for chaotic systems. We observe that $\langle Q_j \rangle = 0$ since (appendix 1)

$$Q_j \equiv \left(\frac{\partial S_j}{\partial \lambda}\right)_x = T_j \left[\left(\frac{\partial E}{\partial \lambda}\right)_x - \frac{1}{T_j}\int_0^{T_j} \mathrm{d}t \frac{\partial H}{\partial \lambda}\right]. \tag{15}$$

Thus

$$\langle Q_j \rangle = T_j \left[\left(\frac{\partial E}{\partial \lambda}\right)_x - \left\langle \frac{\partial H}{\partial \lambda} \right\rangle\right] = 0 \tag{16}$$

where we have used the fact that the long time limit of the time average of a phase space quantity is its average over the entire accessible phase space. The equality with zero follows from (3).

We further assume that Q follows a Gaussian distribution with width $\langle Q^2 \rangle$. This is reasonable if there are no special parameter values where the actions of the periodic orbits change dramatically. Thus

$$\begin{aligned}\left\langle \sin^2\left[\frac{\Lambda Q}{2\hbar}\right]\right\rangle &\approx \frac{1}{\sqrt{2\pi\langle Q^2\rangle}} \int_{-\infty}^{\infty} \mathrm{d}Q \, \sin^2\left[\frac{\Lambda Q}{2\hbar}\right] \exp\left(-\frac{Q^2}{2\langle Q^2\rangle}\right) \\ &= \frac{1}{2}\left[1 - \exp\left(-\frac{\Lambda^2\langle Q^2\rangle}{2\hbar^2}\right)\right].\end{aligned} \tag{17}$$

To calculate $\langle Q^2 \rangle$ we write (15) as

$$\begin{aligned}Q_j &= \int_0^{T_j} \mathrm{d}t \left(\frac{\partial E}{\partial \lambda} - \frac{\partial H}{\partial \lambda}(q_j(t)p_j(t);\lambda)\right) \\ &\equiv \int_0^{T_j} \mathrm{d}t \, B_j(t).\end{aligned} \tag{18}$$

Thus

$$Q_j^2 = \int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2 \, B_j(t_1)B_j(t_2) \tag{19}$$

and

$$\langle Q_j^2 \rangle = 2T \int_0^{\infty} \mathrm{d}t \langle B_j(0)B_j(t)\rangle. \tag{20}$$

Now

$$\begin{aligned}\langle B^2(0)\rangle &= \left\langle \left(\frac{\partial E}{\partial \lambda} - \frac{\partial H}{\partial \lambda}\right)^2 \right\rangle \\ &= \left\langle \left(\frac{\partial H}{\partial \lambda}\right)^2 \right\rangle - \left\langle \frac{\partial H}{\partial \lambda} \right\rangle^2.\end{aligned} \tag{21}$$

For a completely chaotic system, correlations should decay with a characteristic time T_c which is the reciprocal of the Liapunov exponent:

$$\langle B(0)B(t)\rangle \approx \langle B^2(0)\rangle e^{-t/T_c}. \tag{22}$$

Hence, we estimate

$$\langle Q^2\rangle = 2TT_c\left[\left\langle\left(\frac{\partial H}{\partial\lambda}\right)^2\right\rangle - \left\langle\frac{\partial H}{\partial\lambda}\right\rangle^2\right]. \tag{23}$$

Collecting the arguments together, (10) becomes

$$V(\Lambda) = 4\int_0^\infty \frac{dT}{T^2}\left[1-\exp\left(-\frac{\Lambda^2\langle Q^2\rangle}{2\hbar^2}\right)\right]\Phi_l(T) \tag{24}$$

where $\Phi_l(T)$ is the appropriate limiting form of $\Phi(T)$ according to whether T is less than or greater than T_2 and to the antiunitary symmetry class of the system.

Following [3] we introduce a scaled time

$$\tau = \frac{1}{2\pi}\frac{T}{\hbar\langle d\rangle} \tag{25}$$

and write

$$\Phi(T) = \frac{\langle d\rangle\hbar^{2\mu+1}K(\tau)}{2\pi}. \tag{26}$$

$K(\tau)$ is the Fourier transform of the correlation function of the spectral density. Rewriting (23) as $\langle Q^2\rangle = AT$ (24) becomes

$$V(\Lambda) = \frac{1}{\pi^2}\int_0^\infty \frac{d\tau}{\tau^2}K(\tau)\left[1-\exp\left(-\frac{\tau\Lambda^2\pi A\langle d\rangle}{\hbar}\right)\right]. \tag{27}$$

Here $\langle d(E(x;\lambda))\rangle$ is the average density of states at the energy corresponding to unfolded energy x and parameter λ.

Finally, we must interpolate $K(\tau)$ between the regimes (13) and (14), i.e. about $\tau = 1$. An obvious choice is the spectral form factor appropriate to either GOE or GUE matrices. The examples we will consider have time reversal invariance so that the appropriate $K(\tau)$ is [11]

$$K_{\mathrm{GOE}}(\tau) = \begin{cases} 2\tau - \tau\ln(1+2\tau) & (\tau \le 1) \\ 2 - \tau\ln\left(\frac{1+2\tau}{2\tau-1}\right) & (\tau \ge 1) \end{cases} \tag{28}$$

Equation (27), with K replaced by K_{GOE}, is the semiclassical expression for the universal part of the parametric number variance. It depends on the average density of states $\langle d\rangle$, the correlation time T_c and $[\langle(\partial H/\partial\lambda)^2\rangle - \langle\partial H/\partial\lambda\rangle^2]$ which are all in principle calculable from the classical dynamics.

The integration range in (27) should actually start not at 0 but at $\tau_{\min}$ corresponding to the shortest periodic orbit of the system. As a result, equation (27), with K replaced by K_{GOE}, is valid only for

$$\Lambda^2 < \Lambda^2_{\max} = \frac{\hbar}{\tau_{\min}\pi A\langle d\rangle}. \tag{29}$$

For $\Lambda > \Lambda_{\max}$, V is non-universal and saturates to $2\Sigma^2(x)$, as discussed earlier. (As shown in [3],this quantity is of order $\log(1/\hbar) \sim \log x$.)

In appendix 2 we show that (27) vanishes as $-\Lambda^2\log(\Lambda^2)$ as $\Lambda^2 \to 0$, for both the GOE and GUE cases. This limit is universal since it depends upon the properties of the longest periodic orbits.

3. Numerical calculations

To compute V for a particular system, one should obtain the spectrum over a wide range of λ, at fixed x, and simply perform the average. In practice, the range of λ over which it is practical to obtain the spectrum is limited, so that the average in (5) will contain but one or two members for large Λ. In order to overcome this, we in fact average V over a range of x. This is not as crude as it might seem, since in the semiclassical limit the density of states diverges and V has only a slow x dependence. We will return to this point below.

The first model system we consider is the hydrogen atom in a uniform magnetic field. The Hamiltonian, in atomic units, is

$$\hat{H} = \frac{\hat{p}_\rho^2 + \hat{p}_z^2}{2} + \frac{\gamma^2}{8}\hat{\rho}^2 - \frac{1}{\sqrt{\hat{\rho}^2 + \hat{z}^2}} \tag{30}$$

where $\hat{\rho}$ and $\hat{z}$ are cylindrical coordinates, $\hat{p}_\rho$ and $\hat{p}_z$ their respective conjugate momenta and γ is the magnetic field strength. We restrict ourselves to the separable subspace $\hat{p}_\theta = 0$. Here γ is the external parameter. The Hamiltonian (30) has a further classical symmetry. Rescaling

$$p = \gamma^{-1/3}\hat{p} \qquad q = \gamma^{2/3}\hat{q} \qquad t = \gamma\hat{t} \qquad H = \gamma^{-2/3}\hat{H} \tag{31}$$

equation (30) becomes

$$H = \frac{p_\rho^2 + p_z^2}{2} + \frac{\rho^2}{8} - \frac{1}{\sqrt{\rho^2 + z^2}} \tag{32}$$

so that the classical system depends only upon the scaled energy $\epsilon = \gamma^{-2/3}\hat{E}$.

We calculated V from 26 partial spectra corresponding to $2.475 \times 10^{-5} < \gamma < 2.6 \times 10^{-5}$, using the procedure of [12–14].

To produce the spectrum $x_j(\gamma)$ we use (3) and (31) to write

$$x_j(\gamma) = \langle N(\hat{E}_j(\gamma); \gamma\rangle = \gamma^{-2/3}\langle N(\epsilon_j(\gamma)); 1\rangle \tag{33}$$

with the explicit γ dependence of the functional form of the average staircase indicated by the term after the semicolon. For a given ϵ, $N(\epsilon;\gamma) \approx \gamma^{-2/3}\langle N(\epsilon); 1\rangle$; that is we expect $N(\epsilon;\gamma)$, plotted in the $(N, \gamma^{-2/3})$ plane, to scatter about the line through the origin with slope $\langle N(\epsilon); 1\rangle$. Numerically fitting such a line for each $\epsilon_j(\gamma)$, we obtained the x spectrum for levels 615 through 843.

A portion of this spectrum is shown in figure 1. For clarity, the continuous levels are shown—the vertical ticks indicate twice the grid spacing. The effect of the unfolding from E to x is now apparent. The level $x_j(\lambda)$ oscillates about the value j, so that the mean spectral density at fixed λ is unity.

We calculated V for $\Lambda \equiv \delta\gamma$ up to 6×10^{-7} (i.e. half the available range in order to include at least two terms) averaged over various integer values of x. Figure 2 shows the parametric number variance averaged over x values $\{615, 727\}$, the diamonds, and $\{728, 840\}$, the crosses. The general form of the plots is the same, with the high x values lying above the low x values. This reflects the fact that real physical systems are

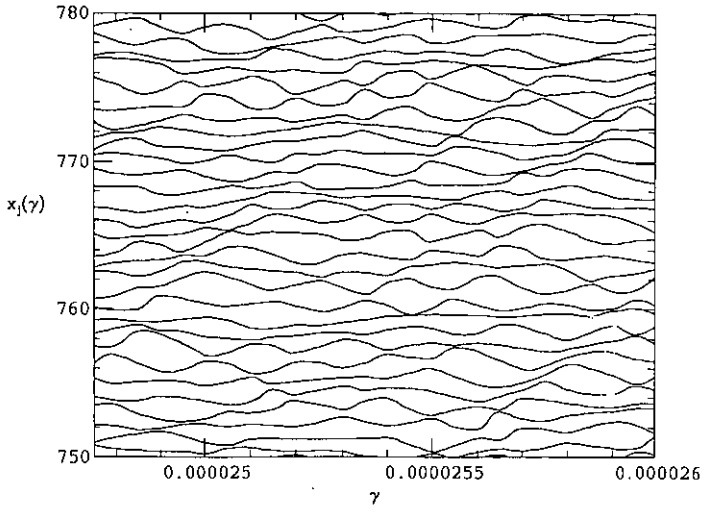

Figure 1. Unfolded spectrum of the hydrogen atom in a constant magnetic field. γ field strength in atomic units, x_j the energy levels normalized by the local average density of states.

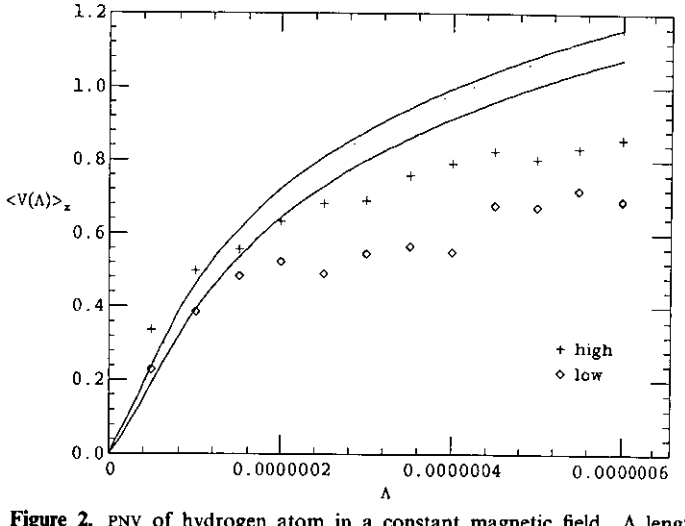

Figure 2. PNV of hydrogen atom in a constant magnetic field. Λ length of γ interval. Diamonds, quantum calculation averaged over levels 615–727; crosses quantum calculation averaged over levels 728–840. Full curves, the semiclassical formula (27) with $\alpha = 750$ higher curve, $\alpha = 500$ lower curve. Dotted curve, semiclassical formula with $\alpha = 696.5$ (see figure 3).

not in fact stationary; the higher lying levels have stronger fluctuations, leading to a larger value of V.

The two full curves of figure 2 are plots of equation (27) treating the quantity $(\pi A\langle d\rangle/\hbar)S^2$ as a single parameter α, chosen arbitrarily to agree with the small Λ points. (The factor $S \equiv 6 \times 10^{-5}/7\pi$ is a rescaling, $\Lambda \to \Lambda' = \Lambda/S$, which makes the integral of (27) easier to compute.) The higher curve is $\alpha = 750$, the lower curve is $\alpha = 500$.

We calculated the factors of α from the dynamics for four different values of x and

the 26 values of γ. The density of states is

$$\langle d(\hat{E}(x))\rangle = \frac{\partial\langle N(\hat{E}(x))\rangle}{\partial\hat{E}} = \gamma^{-2/3}\left.\frac{\mathrm{d}x}{\mathrm{d}\epsilon}\right|_{\epsilon=\gamma^{-2/3}\hat{E}(x)} \tag{34}$$

and we approximate the differential by the difference $(x_{j+1}-x_j)/(\epsilon_{j+1}-\epsilon_j)$ where $x_j < x < x_{j+1}$. The Liapunov exponent is calculated from a chaotic trajectory of appropriate ϵ as described in [15]. The quantity $[\langle(\partial H/\partial\lambda)^2\rangle - \langle\partial H/\partial\lambda\rangle^2]$ is calculated by explicitly averaging $\partial H/\partial\lambda$ over the volume of phase space corresponding to fixed γ and $\hat{E}(x,\gamma)$.

Figure 3 shows α as a function of γ for the four values $x = 743, 771, 799, 827$ which all lie in the high group of levels of figure 2. We see that there are strong fluctuations, but that overall α, and therefore the semiclassical expression (27) for $V(\Lambda)$, increases with x. Averaging over γ and the four x values gives $\alpha = 696.5$, which is the dotted curve of figure 2. There is good agreement with the small Λ values.

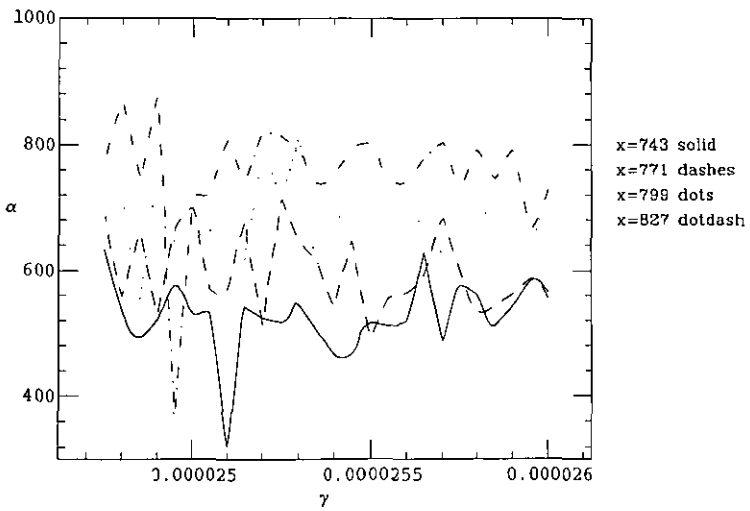

Figure 3. $\alpha(x,\gamma)$ calculated from the dynamics. Full curve, $x = 743$; broken curve, $x = 771$; dotted curave $x = 799$; chain curve $x = 827$. The average value, $\alpha = 696.5$ corresponds to the dotted curve of figure 2.

The most probable reason for the deviation of the semiclassical estimate (27) from the high lying values is that in arriving at (10) we assumed that Λ was small. The saturation reflects the independence of Hamiltonians separated by large Λ. The saturation value should be related to $\Sigma^2(x)$.

The second example we consider is the so called Africa billiard, described in [16], and whose spectrum we calculate after [17].

The boundary of Africa is a conformal map of the unit circle ($z = \mathrm{e}^{\mathrm{i}\theta}$) from $\mathbb{C}$ to $\mathbb{C}$

$$w = z + B_2 z^2 + B_3 z^3 \mathrm{e}^{\mathrm{i}\varphi} \tag{35}$$

where B_2, B_3 and φ are parameters. The Hamiltonian is

$$H = \begin{cases} \frac{1}{2}P^2 & \text{inside billiard} \\ \infty & \text{outside billiard} \end{cases} \tag{36}$$

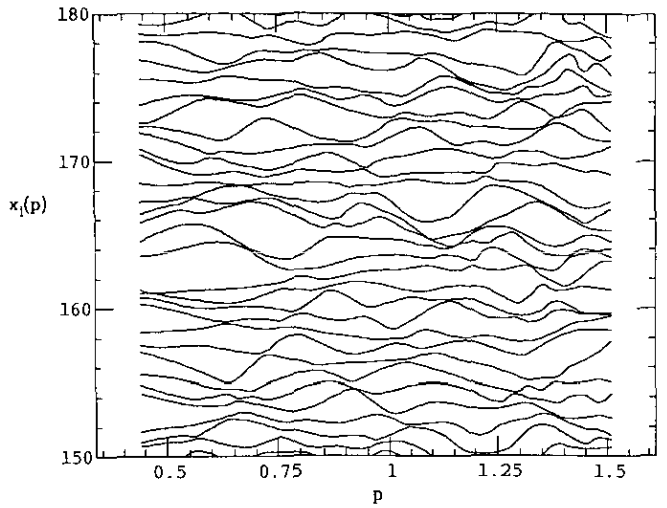

Figure 4. Unfolded spectrum of the Africa billiard with $B_2 = (1/\sqrt{10})\cos p$, $B_3 = (1/\sqrt{15})\sin p$, $\varphi = \pi/3$. p constant area shape parameter. x_j the energy levels normalized by the local average density of states.

with P a dimensionless momentum.

The quantum spectrum (see [17] for full details) is calculated from the basis states of the circular billiard, which are products of Bessel functions and sine or cosine functions. As Africa has (generically) no discrete symmetries, there is full mixing between the sine and cosine subspaces. By means of the mapping (35), the Schrödinger equation corresponding to (36) can be written in the circle basis. This leads to an (infinite) matrix whose eigenvalues are the spectrum.

Only the lowest 20% of the eigenvalues of a finite truncation of this matrix are converged to within 5% of the average spacing. We worked with matrices of size 1000 so that 200 converged energy levels were available. The 5% convergence also determines a minimum parameter increment corresponding to the smallest resolvable level fluctuation.

The average staircase for a smooth billiard is given by the Weyl formula

$$\langle N(E)\rangle = \frac{\mathscr{A}E}{4\pi} - \frac{\mathscr{L}\sqrt{E}}{4\pi} + \frac{1}{6} \tag{37}$$

where $\mathscr{A} = \pi(1 + 2B_2^2 + 3B_3^2)$ is the area and $\mathscr{L}$ the perimeter.

The billiard is described by the three parameters φ, B_2 and B_3. In order to compute the PNV it is necessary to choose a combination of them that can be parametrised by a single quantity. Our choice was to fix φ as $\pi/3$, and $2B_2^2 + 3B_3^2$ as $1/5$, a choice which also fixes the area at $\mathscr{A} = \frac{6}{5}\pi$. The remaining freedom in choosing B_2 and B_3 provides the external parameter with which the PNV is calculated. We denoted it as p and defined it as an angle in the (B_2, B_3) plane:

$$B_2 = \frac{1}{\sqrt{10}}\cos p \qquad B_3 = \frac{1}{\sqrt{15}}\sin p\,. \tag{38}$$

Figure 4 shows a portion of the (x, p) spectrum for $\pi/10 < p < \pi/2$. The figure is qualitatively the same as figure 1; each of the levels oscillates about an average value and the mean level spacing is unity.

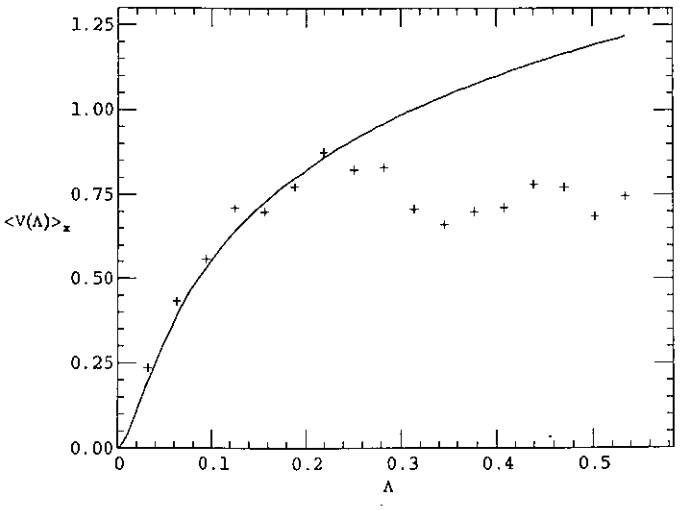

Figure 5. PNV of Africa billiard. Λ length of p interval. Crosses quantum calculation averaged over levels 99–198; full curve, equation (27) with $\alpha = 1000$.

In calculating the parametric number variance we restricted ourselves to the interval $0.14\pi < p < 0.48\pi$ in order to ensure that there were no correlations due to symmetries, for there is a twofold symmetry when $p = 0$ ($B_3 = 0$) and a fourfold symmetry when $p = \pi/2$.

Figure 5 shows the calculated values of V averaged over $99 < x < 198$. The line is (27) with $\alpha \equiv \pi \mathscr{A} \langle d \rangle / \hbar = 1000$. For this value of α, the small Λ behaviour of appendix 2 is only evident for p increments smaller than the minimum step we could resolve—0.01π.

For the Africa billiard

$$\alpha = \frac{\pi}{32} \frac{\mathscr{L}^2}{\mathscr{A}^2} \frac{D^3}{\wp} (1 - D^{-1/2}) I \tag{39}$$

where $\wp$ is the discrete time Liapunov exponent

$$I = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r \frac{r}{|\partial w/\partial z|^2} \left(\frac{\partial |\partial w/\partial z|^2}{\partial \lambda} \right)^2 \tag{40}$$

and

$$D = 1 + \left(1 + \frac{16\pi(x - \frac{1}{6}).\mathscr{A}}{\mathscr{L}^2} \right)^{1/2}$$

is related to the dimensionless energy obtained by inverting (37). The energy dependence of α is thus $\alpha \approx D^3 \approx x^{3/2}$ for large x.

We remark that $\langle \partial H/\partial \lambda \rangle = 0$, I being the dimensionless product of $\langle d \rangle$ and $\langle (\partial H/\partial \lambda)^2 \rangle$. Moreover, we calculate $\langle (\partial H/\partial \lambda)^2 \rangle$ not in the w plane, where the discontinuities in H give infinities, but in the z plane, where the metric $|\partial w/\partial z|^2$ is determined by the conformal map.

The value of α determined from (39) is 10 000, an order of magnitude too large. We suspect that the crude estimate (22) for $\langle B(0)B(t)\rangle$ accounts for the discrepancy. We have shown that the α which empirically fits averages done over small groups of 20 x values does increase as $x^{2/3}$.

According to (6) we can obtain the saturated value of V from $\Sigma^2(x)$. We calculated $\Sigma^2(x)$, averaged over nine values of p spaced at $\Lambda = 0.04\pi$, and $99 < x < 198$. The resulting $\bar{\Sigma}^2$ value of 0.36 is indeed half the saturation value, supporting the claim that 'distant' spectra are essentially uncorrelated.

The parametric number variance is a new diagnostic tool which is complementary to the established spectral measures. These known statistics use large complete portions of a spectrum to characterize the dynamics. In contrast, the PNV requires only a small complete portion of the spectrum, albeit over a large parameter range. In experiments, the PNV may often be more accessible to measurement, since it can be more difficult to prepare the system in many different quantum states than to vary the external perturbation.

The semiclassical theory of the PNV shows excellent qualitative agreement with the quantum calculations. In particular, the choice of the GUE form factor in (27) leads to curves that do not fit the GOE data presented here for any choice of α. This strongly suggests that the PNV can discriminate between GOE and GUE spectra.

Acknowledgments

We thank R Mondragon for guidance in computations with Africas. Support for this work was provided by the Pulver Scholarship fund (JG) and the United States–Israel Binational Science Foundation (US). MVB thanks the Weizmann Institute for its hospitality while this work was carried out.

Appendix 1.

We prove relation (15). A periodic orbit may be thought of as a mapping, labelled by the continuous parameter λ, from the unit circle into phase space. Thus, the action associated with the orbit is

$$S(\lambda) = \int_0^{2\pi} p(\theta,\lambda)\frac{\partial q(\theta,\lambda)}{\partial\theta}\mathrm{d}\theta. \tag{A1.1}$$

Similarly, the physical time is $t = t(\theta,\lambda)$.

Now

$$\begin{aligned}\frac{\partial S}{\partial\lambda} &= \int_0^{2\pi}\left[\frac{\partial p}{\partial\lambda}\frac{\partial q}{\partial\theta} + p\frac{\partial^2 q}{\partial\theta\partial\lambda}\right]\mathrm{d}\theta \\ &= \int_0^{2\pi}\left[\frac{\partial p}{\partial\lambda}\frac{\partial q}{\partial\theta} - \frac{\partial p}{\partial\theta}\frac{\partial q}{\partial\lambda}\right]\mathrm{d}\theta\end{aligned} \tag{A1.2}$$

after an integration by parts.

The time measured along the periodic orbit is monotonic in θ so we may write the equations of motion as

$$\frac{\partial H}{\partial p} = \dot{q} = \frac{\partial q}{\partial\theta}\frac{\partial\theta}{\partial t} \qquad -\frac{\partial H}{\partial q} = \dot{p} = \frac{\partial p}{\partial\theta}\frac{\partial\theta}{\partial t} \tag{A1.3}$$

so that

$$\frac{\partial S}{\partial \lambda} = \int_0^{2\pi} \left(\frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} + \frac{\partial H}{\partial q}\frac{\partial q}{\partial \lambda} \right) \frac{\partial t}{\partial \theta} \mathrm{d}\theta$$
$$= \int_0^T \left(\frac{\mathrm{d}H}{\mathrm{d}\lambda} - \frac{\partial H}{\partial \lambda} \right) \mathrm{d}t$$
$$= T \left[\left(\frac{\partial E}{\partial \lambda} \right) - \frac{1}{T} \int_0^T \mathrm{d}t \frac{\partial H}{\partial \lambda} \right]. \tag{A1.4}$$

Appendix 2.

We derive the small parameter behaviour of (27), rewritten as

$$V(u) = \frac{1}{\pi^2} \int_0^1 \frac{\mathrm{d}t}{t^2} K_1(t)(1 - \mathrm{e}^{-ut}) + \frac{1}{\pi^2} \int_1^\infty \frac{\mathrm{d}t}{t^2} K_2(t)(1 - \mathrm{e}^{-ut}) \tag{A2.1}$$

where K_1 and K_2 are given by (28) for GOE-type systems, while for GUE-type systems

$$K_1(t) = t \qquad t \leq 1$$
$$K_2(t) = 1 \qquad t \geq 1. \tag{A2.2}$$

We first observe that $K_2(t)$ is constant in the limit that $t \to \infty$. Hence, we choose an $\alpha > 1$ such that for $t > \alpha$, $K(t)$ in (A2.1) may be replaced by the appropriate constant.

In the small u limit $\frac{1}{u} \gg \alpha$. Thus, for $0 < t < \alpha$, $(1 - \mathrm{e}^{-ut})/t \simeq u$. The small u limit of $V(u)$ is therefore

$$V(u) \simeq u \int_0^\alpha \mathrm{d}t \frac{K(t)}{t} + \int_\alpha^\infty \mathrm{d}t \frac{(1 - \mathrm{e}^{-ut})}{t^2}. \tag{A2.3}$$

The first term vanishes linearly with u. The second term, after the substitution $x = ut$ and an integration by parts, becomes

$$\frac{1}{\alpha}(1 - \mathrm{e}^{-u\alpha}) + u \int_{\alpha u}^\infty \frac{\mathrm{d}x}{x} e^{-x}.$$

Again, the first of these terms vanishes linearly with u. The second term is evaluated by breaking up the integral into two pieces at an arbitrary but fixed value β such that $\alpha u < \beta < \infty$. The integral from β to ∞ is independent of u, so this gives another term that vanishes linearly with u.

The remaining integral satisfies

$$u\mathrm{e}^{-\alpha u} \int_{\alpha u}^\beta \frac{\mathrm{d}x}{x} > u \int_{\alpha u}^\beta \mathrm{e}^{-x} \frac{\mathrm{d}x}{x} > u\mathrm{e}^{-\beta} \int_{\alpha u}^\beta \frac{\mathrm{d}x}{x}$$

and thus vanishes as $-u\log(u)$, which is slower than linear.

References

[1] Bohigas O, Giannoni M J and Schmidt C 1986 *Quantum Chaos and Statistical Nuclear Physics (Lecture Notes in Physics* **263***)* ed T H Seligman and H Nishioka (Berlin: Springer) p 18

[2] Berry M V and Tabor M 1977 *Proc. R. Soc. Lond.* A **356** 375
[3] Berry M V 1985 *Proc. R. Soc. Lond.* A **400** 229
[4] Zaslavskii G M 1979 *Sov. Phys. Usp.* **22** 788
[5] Pandey A 1979 *Ann. Phys.* **119** 170
[6] Bohigas O and Giannoni M J 1984 *Mathematical and Computational Methods in Nuclear Physics (Lecture Notes in Physics* **209***)* ed J S Dehesa *et al* (Berlin: Springer) p 1
[7] Blümel R, Goldberg J and Smilansky U 1988 *Z. Phys.* D **9** 95
[8] Wilkinson M 1988 *J. Phys. A: Math. Gen.* **21** 4021
[9] Gutzwiller M C 1971 *J. Math. Phys.* **12** 343
[10] Hannay J H and Ozorio De Almeida A M 1984 *J. Phys. A: Math. Gen.* **17** 3429
[11] Mehta M L 1967 *Random Matrices and the Statistical Theory of Energy Levels* (New York: Academic)
[12] Wunner G and Ruder H 1987 *Phys. Scri.* **36** 291
[13] Wunner G, Zeller G, Woelk U, Schweizer W, Niemeier R, Geyer F, Friedrich H and Ruder H 1988 *Atomic Spectra and Collisions in External Fields II* ed. K T Taylor, C W Clark and M H Nayfeh (New York: Plenum)
[14] Wunner G, Schweizer W and Ruder H 1988 *The Hydrogen Atom Symposium* ed F Bassani *et al* (Berlin: Springer)
[15] Schweizer W, Niemeier R, Friedrich H, Wunner G and Ruder H 1988 *Phys. Rev.* A **38** 1724
[16] Berry M V 1987 *J. Phys. A: Math. Gen.* **20** 2389
[17] Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 1049