

The adiabatic phase and Pancharatnam's phase for polarized light

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Abstract. In 1955 Pancharatnam showed that a cyclic change in the state of polarization of light is accompanied by a phase shift determined by the geometry of the cycle as represented on the Poincaré sphere. The phase owes its existence to the non-transitivity of Pancharatnam's connection between different states of polarization. Using the algebra of spinors and 2×2 Hermitian matrices, the precise relation is established between Pancharatnam's phase and the recently discovered phase change for slowly cycled quantum systems. The polarization phase is an optical analogue of the Aharonov–Bohm effect. For slow changes of polarization, the connection leading to the phase is derived from Maxwell's equations for a twisted dielectric. Pancharatnam's phase is contrasted with the phase change of circularly polarized light whose direction is cycled (e.g. when guided in a coiled optical fibre).

1. Introduction

In a remarkable paper published thirty years ago, Pancharatnam ([1], reprinted in [2]) considered the phase of a beam of light whose state of polarization is made to change. His central result, when expressed symmetrically, concerned a beam that is returned to its original state of polarization via two intermediate polarizations. He showed that the phase does not return to its original value but increases by $-\Omega/2$ where Ω is the area (solid angle) spanned on the Poincaré sphere [3, 4] by the geodesic triangle whose vertices are the three polarizations. Ramaseshan and Nityananda [5] point out that Pancharatnam's study of classical light anticipated a recent general extension [6] of the adiabatic theorem of quantum mechanics, giving the phase change in the state of a system whose environmental parameters are taken slowly round a cycle. My intention here is to bring out the full originality of Pancharatnam's contribution by expressing his optics in quantum-mechanical language and clarifying the relation between his phase and the adiabatic phase.

First, it is necessary to recall the description of polarization in terms of the Poincaré sphere. For a monochromatic wave travelling in the direction z the electric displacement vector lies in the xy plane, and its state of polarization is described by the complex unit vector

$$\mathbf{d} = (d_x, d_y), \quad (\mathbf{d}^* \cdot \mathbf{d} = 1). \quad (1)$$

This is equivalent to a two-component spinor

$$|\psi\rangle = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}, \quad \psi_{\pm} \equiv (d_x \pm id_y)/\sqrt{2}. \quad (2)$$

Thus $\langle\psi|\psi\rangle=1$. Each such $|\psi\rangle$ is the eigenvector with eigenvalue $+1/2$ of some 2×2 Hermitian matrix (the polarization matrix) of the form

$$\mathbf{H}(\mathbf{r})=\mathbf{r}\cdot\boldsymbol{\sigma}=\frac{1}{2}\begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix}=\frac{1}{2}\begin{bmatrix} \cos\theta & \sin\theta\exp(-i\phi) \\ \sin\theta\exp(i\phi) & -\cos\theta \end{bmatrix} \quad (3)$$

where $\boldsymbol{\sigma}$ is the vector of Pauli spin matrices and $\mathbf{r}=(x,y,z)$ is a unit vector with polar angles θ and ϕ . (\mathbf{H} is the matrix representing the operator $|\psi\rangle\langle\psi|-1/2$.)

The Poincaré sphere is the sphere with coordinates θ, ϕ . Each position \mathbf{r}_A defines a matrix \mathbf{H} , whose eigenvector is the polarization $|\psi(\mathbf{r}_A)\rangle$ which will henceforth be written simply as $|A\rangle$. Thus

$$\mathbf{H}(\mathbf{r}_A)|A\rangle=+\frac{1}{2}|A\rangle. \quad (4)$$

The poles $\theta=0$ and $\theta=\pi$ have $\psi_-=0$ and $\psi_+=0$ respectively, and represent the two senses of circular polarization. Points on the equator $\theta=\pi/2$ have $|\psi_+|=|\psi_-|$ and represent the different direction of linear polarization.

2. Pancharatnam's connection

The phase difference between two distinct polarizations $|A\rangle$ and $|B\rangle$ is not determined by the eigenequation (4). The first of Pancharatnam's two main contributions was to point out that a natural convention is to measure the phase difference from a reference condition in which interference of the superposed beams $|A\rangle$ and $|B\rangle$ gives maximum intensity. This intensity is

$$\langle(A|+|B|)(|A\rangle+|B\rangle)=2+2|\langle A|B\rangle|\cos(\text{ph}\langle A|B\rangle) \quad (5)$$

Thus the phase difference of $|A\rangle$ and $|B\rangle$ is simply the phase, $\text{ph}\langle A|B\rangle$, of their scalar product. The reference condition, in which $|A\rangle$ and $|B\rangle$ are defined as being in phase, corresponds to $\langle A|B\rangle$ real and positive, and will be called Pancharatnam's connection. It defines a relation between any two (non-orthogonal) states, not just neighbouring ones, and so resembles the geometric notion of distant parallelism.

Pancharatnam's second contribution was to show that his connection is non-transitive: if $|B\rangle$ is in phase with $|A\rangle$, and $|C\rangle$ with $|B\rangle$, then $|C\rangle$ need not be in phase with $|A\rangle$. Indeed if $|C\rangle$ is in phase with a state $|A'\rangle$ corresponding to the point \mathbf{r}_A on the sphere, then

$$\langle A|A'\rangle=\exp(-i\Omega_{ABC}/2), \quad (6)$$

where Ω_{ABC} is the solid angle of the geodesic triangle ABC on the sphere.

To relate Pancharatnam's connection and its consequence (6) to the quantum adiabatic phase, consider a two-state quantum system (e.g. a spin-1/2 particle) whose evolution is governed by the Hamiltonian operator (3), in which \mathbf{r} is driven slowly round a circuit C on the sphere. Adiabatic theory based on the Schrödinger equation then shows [6] that if the system is initially in the eigenstate $|A\rangle$ with (say) the greater energy, it will remain in that eigenstate $|\psi[\mathbf{r}(t)]\rangle$ with a phase given (after subtraction of a trivial dynamical contribution) by the connection

$$\langle\psi|\partial|\psi\rangle/\partial t=0. \quad (7)$$

This connection is mathematically natural [7-9]. For the two-state system it follows from (7) that at the end of the circuit the state $|A'\rangle$ is phase-shifted relative to the initial one, the relation being

$$\langle A|A'\rangle=\exp[-i\Omega(C)/2] \quad (8)$$

where $\Omega(C)$ is the solid angle of the circuit C.

The close similarity between (6) and (8) suggests that Pancharatnam's connection is equivalent to the adiabatic connection (7). Thus if $|\psi\rangle$ is continued with the aid of (7) along the shorter geodesic arc from A to B, that is

$$|\psi(0)\rangle = |A\rangle, \quad |\psi(T)\rangle = |B\rangle \tag{9}$$

then $\langle A|B\rangle$ must be real and positive.

To show this equivalence, we introduce the unitary operator taking $|A\rangle$ and $|B\rangle$:

$$|B\rangle = U|A\rangle \tag{10}$$

For the connection (7) the operator is [10]

$$U = T \exp \left[-i \int_0^T dt' \omega(t') \cdot \sigma \right] \tag{11}$$

where T denotes time-ordering and $\omega(t)$ is the instantaneous angular velocity of a frame being parallel-transported on the sphere along the path joining A to B. For a geodesic arc,

$$\omega(t) = \omega(t)\mathbf{n}, \tag{12}$$

where \mathbf{n} is the constant unit vector perpendicular to the plane of the arc (figure 1). If the angle between A and B is $\int_0^T dt \omega(t) = \phi_{AB}$, (11) becomes

$$\begin{aligned} U &= \exp(-i\phi_{AB}\mathbf{n} \cdot \sigma) \\ &= \cos(\phi_{AB}/2) - i \sin(\phi_{AB}/2)\mathbf{n} \cdot \sigma. \end{aligned} \tag{13}$$

Thus (10) gives

$$\langle A|B\rangle = \cos(\phi_{AB}/2) - i \sin(\phi_{AB}/2) \langle A|\mathbf{n} \cdot \sigma|A\rangle. \tag{14}$$

The second term vanishes because it involves the expectation value of the spin component along a direction (\mathbf{n}) perpendicular to that for which $|A\rangle$ is an eigenstate (i.e. $\mathbf{n} \cdot \mathbf{r}_A = 0$). Thus $\langle A|B\rangle = \cos(\phi_{AB}/2)$, which is indeed real and positive if $|\phi_{AB}| < \pi$, that is for the shorter of the geodesic arcs connecting A and B.

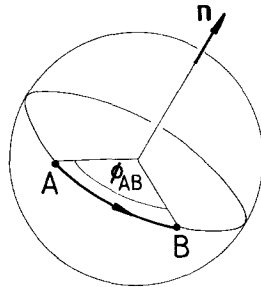


Figure 1. Geodesic arc on the Poincaré sphere.

3. Aharonov-Bohm effect on the Poincaré sphere

The solid-angle formulae (6) and (8) depend only on the connection (7), independently of its origin in adiabatic theory. Therefore the polarization changes need not be slow; they can be accomplished suddenly, for example with birefringent crystal analysers. This is how Pancharatnam [2] confirmed his theory experimentally; the phase shifts were detected by interference.

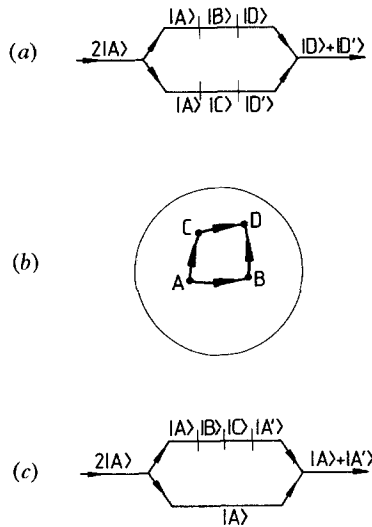


Figure 2. Aharonov-Bohm effect with polarized light: (a) different histories of two beams; (b) equivalent circuit on Poincaré sphere; and (c) alternative scheme, directly implementing equation (6).

It follows from (6) that if a beam $|A\rangle$ is divided into two beams (figure 2 (a)) which pass through different intermediate states $|B\rangle$ and $|C\rangle$ and are then analysed into a polarization D, their states $|D\rangle$ and $|D'\rangle$ will differ in phase by half the solid angle of the geodesic polygon ABCDA on the sphere (figure 2 (b)). (The polarizers are here assumed to produce the changes of state according to Pancharatnam's connection. In practice there will be extra phase shifts of refractive origin, but these are easy to compensate.)

This result is precisely analogous to the phase shift later predicted by Aharonov and Bohm [11, 12], according to which two electron beams develop a phase shift proportional to the magnetic flux they enclose. For polarized light the analogue of magnetic flux is the solid angle of the polygon on the Poincaré sphere. This is also a flux, namely that of an abstract monopole [6] of strength $-1/2$, situated at the centre of the sphere.

There is a simpler geometry, more directly implementing equation (6), in which one beam continues in the state $|A\rangle$ while the other returns to the same polarization A in state $|A'\rangle$ via intermediate states $|B\rangle$ and $|C\rangle$ (figure 2 (c)); interference between $|A\rangle$ and $|A'\rangle$ reveals the phase difference $-\Omega_{ABC}/2$. A special case, at the same time both trivial and instructive, is when A, B and C lie equally spaced on the equator of the sphere and so correspond to linear polarizations successively rotated by $\pi/3$. The geodesic triangle ABC is a hemisphere, with solid angle 2π , so that the phase shift between $|A\rangle$ and $|A'\rangle$ is π , as it must be because the effect of the three rotations is to reverse the field (1). This example resolves the paradox of polarization being governed by the same algebra as quantum states representing spin-1/2 particles, which change sign in a 2π rotation, even though photons are quantum particles with spin 1, which do not. The explanation is that a complete rotation of the spin-1/2 matrix (3) (here accomplished by increasing ϕ by 2π , with $\theta = \pi/2$) corresponds to only a half turn of the photon, that is, to a reversal of the field.

A related result of Pancharatnam [1] which can alternatively be derived and interpreted with spinors concerns the decomposition of a beam with polarization C into two beams with polarizations A and B: the relative phase Δ_{AB} of the two beams is $\pi - \Omega_{C'BA}/2$, where C' is the antipode of C. To derive this we write the spinor decomposition as

$$|C\rangle = a|A\rangle + b|B\rangle, \tag{15}$$

and choose $|A\rangle$ and $|B\rangle$ to be in phase so that Δ_{AB} is the phase of b/a . Now the antipodal state $|C'\rangle$ is orthogonal to $|C\rangle$, so that

$$b/a = -\langle C'|A\rangle/\langle C'|B\rangle. \tag{16}$$

We can choose $|C'\rangle$ to be in phase with $|A\rangle$, so that $\langle C'|A\rangle$ is real and positive. Then

$$\Delta_{AB} = \pi - \lambda \tag{17}$$

where λ is the phase of $\langle C'|B\rangle$. Now consider the triangle BAC'B, starting in state $|B\rangle$ and ending in state $|B'\rangle$ which is in phase with $|C'\rangle$. From (6), $\langle B|B'\rangle = \exp(-i\Omega_{C'BA}/2)$, so that

$$\lambda = +\Omega_{C'BA}/2 \tag{18}$$

which with (17) is the required result.

4. Adiabatic changes of polarization

I have suggested elsewhere [13] that changes in the state of polarization can be accomplished continuously, rather than discretely, with a transparent birefringent gyrotropic medium twisted along the beam so that its dielectric tensor varies through a cycle. If the dielectric variation is slow enough, the beam remains in a polarization state determined by the local uniform medium: the displacement unit vector $\mathbf{d}(z)$ is one of the two eigenvectors of the 2×2 Hermitian submatrix $\mathbf{M}(z)$ consisting of the components transverse to z of the inverse dielectric tensor [14].

As any 2×2 Hermitian matrix can be expressed in the form (3) plus a multiple of the unit matrix, a cyclic change in $\mathbf{M}(z)$ corresponds to a circuit on the Poincaré sphere. The circuit will be accompanied by a phase factor of the form (8), provided the local eigenstates are continued by means of the analogue of (7), namely

$$\mathbf{d}^*(z) \cdot \mathbf{d}'(z) = 0, \tag{19}$$

where the prime denotes differentiation with respect to z . This connection cannot be justified by invoking the quantum adiabatic result, because this originates in the time-dependent Schrödinger equation, whereas these fields are governed by Maxwell's equations for time-independent (i.e. harmonic) waves. In this section I will outline the derivation of (19).

For waves with frequency w , Maxwell's equations reduce to the following equation for the electric displacement $\mathbf{D}(z) \exp(-iwt)$:

$$(\mathbf{M} \cdot \mathbf{D})' + k^2 \mathbf{D} = 0, \tag{20}$$

where $k = w/c$. For slowly varying $\mathbf{M}(z)$ each of the two forward-propagating adiabatic solutions has the form

$$\mathbf{D}(z) \approx a(z) \mathbf{d}(z) \exp \left[ik \int_0^z dz' n(z') \right] \tag{21}$$

where a is a real amplitude and n is the refractive index corresponding to one of the normalized eigenpolarizations \mathbf{d} , satisfying

$$\mathbf{M} \cdot \mathbf{d} = n^{-2} \mathbf{d}. \quad (22)$$

The connection satisfied by \mathbf{d} , and the form of $a(z)$, are determined by (20) in the adiabatic limit of large k (relative to the rate at which \mathbf{M} varies). The results can however be obtained more simply from the vanishing divergence of the Poynting vector for monochromatic waves. In the present case this implies the constancy of

$$P = \text{Im } \mathbf{D}^* \cdot \mathbf{M} \cdot (\mathbf{M} \cdot \mathbf{D})', \quad (23)$$

which can also be derived directly from (20) using the Hermiticity of \mathbf{M} . Substitution of the *ansatz* (21) and use of (22) gives

$$P = ka^2/n^3 + (a^2/n^4) \text{Im } \mathbf{d}^* \cdot \mathbf{d}'. \quad (24)$$

Constancy of the term involving k gives

$$a(z) = \text{constant} \times [n(z)]^{3/2}. \quad (25)$$

Constancy of the term independent of k now implies that $\text{Im } \mathbf{d}^* \cdot \mathbf{d}'$ is proportional to $n(z)$, and the constant of proportionality can be identified as zero by considering stretches of the beam along which \mathbf{M} is constant, for which (21) is exact with \mathbf{d} (and a) constant. Thus $\text{Im } \mathbf{d}^* \cdot \mathbf{d}' = 0$, and we deduce (19) from $\text{Re } \mathbf{d}^* \cdot \mathbf{d}' = 0$ which follows from normalization.

The exact solution of (20) can be written as the sum of four terms of the form (21) with complex amplitudes $a(z)$, representing forward and backward waves with each of the two local eigenpolarizations. It is not hard to show that transitions between these waves (that is, violations of adiabaticity) are exponentially weak if $\mathbf{M}(z)$ varies analytically, the strongest being between the two waves travelling in the same direction and having amplitudes of order $\exp(-kL\Delta n)$, where Δn is the refractive-index difference between the two polarizations and L the length scale over which \mathbf{M} varies.

5. Concluding remarks

Pancharatnam's phase (6) should not be confused with another optical phase given by a solid angle, namely that predicted [15] and found ([16], see also [17, 18]) in circularly polarized light propagating along a twisting path (for example by being guided in a coiled optical fibre).

In (6) the solid angle refers to a path on the Poincaré sphere representing a two-state system. The photons can be regarded as having only two states, in spite of being particles with spin 1, because the light is always travelling in the same direction and for plane waves transversality permits only helicity states, in which the spin is parallel or antiparallel to the propagation direction \mathbf{k} .

In the coiled light experiments, on the other hand, the direction of the light is made to change, and the solid angle refers to a path on the sphere of directions \mathbf{k} . Such a path represents the rotation of a full three-component spinor reflecting the unit (rather than 1/2) spin of the photon, because only in a moving frame whose z axis coincides with \mathbf{k} does the zero-spin (longitudinal) spinor component vanish. In a

fixed frame the spinor representing transverse light in direction \mathbf{k} with polar angles θ and ϕ has the general form

$$|\psi\rangle = \frac{1}{2} \begin{bmatrix} [\alpha + \beta + (\alpha - \beta) \cos \theta] \exp(-i\phi) \\ \sqrt{2}(\alpha - \beta) \sin \theta \\ [\alpha + \beta - (\alpha - \beta) \cos \theta] \exp(i\phi) \end{bmatrix}, \quad (26)$$

where $|\alpha|^2 + |\beta|^2 = 1$; this involves all three components. Moreover, it is only in the adiabatic limit of slow coiling that the light is exactly transverse: for any finite rate of rotation of \mathbf{k} there are non-zero longitudinal components, associated with weak non-adiabatic transitions between the two polarizations [17].

The difference between the Poincaré sphere (spin 1/2) and the sphere of directions (spin 1) is reflected in the different dependence of the phase shift on the solid angle Ω . For a polarization cycle the phase change is $-\Omega/2$, whereas for a direction cycle the phases for the two circular polarizations are $\pm\Omega$.

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References

- [1] PANCHARATNAM, S., 1956, *Proc. Ind. Acad. Sci. A*, **44**, 247–262.
- [2] PANCHARATNAM, S., 1975, *Collected Works of S. Pancharatnam* (Oxford: University Press).
- [3] BORN, M., and WOLF, E., 1959, *Principles of Optics* (London: Pergamon).
- [4] CLARKE, D., and GRAINGER, J. F., 1971, *Polarized Light and Optical Measurement* (Oxford: Pergamon).
- [5] RAMASESHAN, S., and NITYANANDA, R., 1986, *Current Science, India*, **55**, 1225–1226.
- [6] BERRY, M. V., 1984, *Proc. R. Soc., Lond. A*, **392**, 45–57.
- [7] SIMON, B., 1983, *Phys. Rev. Lett.*, **51**, 2167–2170.
- [8] AHARONOV, Y., and ANANDAN, J., 1987, *Phys. Rev. Lett.*, **58**, 1593–1596.
- [9] SEGERT, J., 1987, *Phys. Rev. A*, **36**, 10–16.
- [10] BERRY, M. V., 1987, *Proc. R. Soc., Lond.*, (to be published).
- [11] AHARONOV, Y., and BOHM, D., 1959, *Phys. Rev.*, **115**, 485–491.
- [12] OLARIU, S., and POPESCU, I., 1985, *Rev. mod. Phys.*, **57**, 339–436.
- [13] BERRY, M. V., 1986, *Fundamental Aspects of Quantum Theory, NATO ASI series Vol. 144*, edited by V. Gorini and A. Frigerio (New York: Plenum), pp. 267–278.
- [14] LANDAU, L. D., LIFSHITZ, E. M., and PITAEVSKII, L. P., 1984, *Electrodynamics of Continuous Media* (second edition) (Pergamon: Oxford).
- [15] CHIAO, R. Y., and WU, Y. S., 1986, *Phys. Rev. Lett.*, **57**, 933–936.
- [16] TOMITA, A., and CHIAO, R. Y., 1986, *Phys. Rev. Lett.*, **57**, 937–940.
- [17] BERRY, M. V., 1987, *Nature, Lond.*, **326**, 277–278.
- [18] KITANO, M., YABUZAKI, T., and OGAWA, T., 1987, *Phys. Rev. Lett.*, **58**, 523.