

rspa.royalsocietypublishing.orgM. V. Berry¹ and Pragya Shukla²

Research



Cite this article: Berry MV, Shukla P. 2015
Hamiltonian curl forces. *Proc. R. Soc. A* **471**:
20150002.
<http://dx.doi.org/10.1098/rspa.2015.0002>

Received: 1 January 2015

Accepted: 19 February 2015

Subject Areas:

mathematical physics

Keywords:Newtonian, follower force, circulatory force,
non-Hamiltonian**Author for correspondence:**

M. V. Berry

e-mail: asymptotic@bristol.ac.uk¹H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK²Department of Physics, Indian Institute of Science, Kharagpur, India

Newtonian forces depending only on position but which are non-conservative, i.e. whose curl is not zero, are termed ‘curl forces’. They are non-dissipative, but cannot be generated by a Hamiltonian of the familiar isotropic kinetic energy + scalar potential type. Nevertheless, a large class of such non-conservative forces (though not all) can be generated from Hamiltonians of a special type, in which kinetic energy is an anisotropic quadratic function of momentum. Examples include all linear curl forces, some azimuthal and radial forces, and some shear forces. Included are forces exerted on electrons in semiconductors, and on small particles by monochromatic light near an optical vortex. Curl forces imply restrictions on the geometry of periodic orbits, and non-conservation of Poincaré’s integral invariant. Some fundamental questions remain, for example: how does curl dynamics generated by a Hamiltonian differ from dynamics under curl forces that are not Hamiltonian?

1. Introduction

In recent papers [1,2], we studied the dynamics of a particle moving under the influence of Newtonian forces that depend on position but not velocity, and whose curl is non-zero. Such ‘curl forces’ are non-conservative: they cannot be represented as the gradient of a scalar potential. The dynamical systems under study can be represented as follows:

$$\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}), \quad \nabla \times \mathbf{F}(\mathbf{r}) \neq 0 \Rightarrow \mathbf{F}(\mathbf{r}) \neq -\nabla U(\mathbf{r}), \quad (1.1)$$

where for the sake of simplicity the mass has been scaled to unity.

There are several reasons for studying curl forces. First, because the link between symmetries and conservation laws, via Noether’s theorem, is broken [1]. Second, because they describe some of the forces that light exerts on small particles [2–6] (notwithstanding a controversy about the use of curl forces as models in engineering mechanics [7]). Third, because although curl



An invited article to commemorate
350 years of scientific publishing at
the Royal Society.

forces are non-conservative, meaning that the work done on a moving particle depends on the path, they are non-dissipative, because volume in the position–velocity space is conserved [1]. Fourth, because curl forces form an interesting subclass within the much-studied class of reversible dynamical systems [8], for which reversing the velocity at some instant causes the particle to retrace its trajectory: curl forces are velocity-independent reversible forces. (Non-reversible forces, such as magnetism and viscosity, involve odd powers of the velocity.)

It is obvious that curl forces cannot arise from a Hamiltonian of the familiar form

$$H = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} + U(\mathbf{r}). \quad (1.2)$$

Our purpose here is to demonstrate that even though curl forces are non-conservative, a large class of them can be generated by Hamiltonians; this is described in §2. Curl forces arising in this way fall into several classes, described in §3. Implications for the geometry of periodic orbits, and for the analogues of Poincaré’s circuit integrals, are discussed in §4. Fundamental unsolved problems remain; we list some of them in §5. Two technical appendices deal with the conditions for forces generated by a Hamiltonian to be independent of velocity, and periodic orbits in linear curl forces.

We choose to employ the Hamiltonian formalism, but the theory could equally be expressed using Lagrangians.

2. Anisotropic kinetic energy

It will suffice to consider particles moving in the plane (x, y) ; we will also use polar coordinates (r, ϕ) , with corresponding unit vector directions $(\mathbf{e}_r, \mathbf{e}_\phi)$. If forces depending only on position are to be generated by a Hamiltonian, this imposes restrictions on its form. In appendix A, we show under wide assumptions that the permitted class is kinetically quadratic and anisotropic, that is

$$H = \frac{1}{2}\alpha p_x^2 + \beta p_x p_y + \frac{1}{2}\gamma p_y^2 + U(x, y). \quad (2.1)$$

(We do not include terms linear in p_x and p_y because these simply generate velocity shifts, and for convenience we retain the β term, although this can be eliminated by a rotation). A much-studied physical example is the anisotropic Kepler problem [9,10], representing motion of a conduction electron near a screened nucleus in a semiconductor; the anisotropy arises because in a crystal the effective mass is a tensor not a scalar.

From Hamilton’s first equation, the velocities $\mathbf{v} \equiv \dot{\mathbf{r}}$ are

$$v_x = \alpha p_x + \beta p_y, \quad v_y = \beta p_x + \gamma p_y. \quad (2.2)$$

The accelerations—that is, the Newtonian forces—are given by Hamilton’s second equation

$$\left. \begin{aligned} F_x &\equiv \ddot{x} = \alpha \dot{p}_x + \beta \dot{p}_y = -(\alpha \partial_x U + \beta \partial_y U) \\ F_y &\equiv \ddot{y} = \beta \dot{p}_x + \gamma \dot{p}_y = -(\beta \partial_x U + \gamma \partial_y U). \end{aligned} \right\} \quad (2.3)$$

and

In vector-matrix form,

$$\mathbf{F}(\mathbf{r}) = -\mathbf{M}\nabla U(\mathbf{r}), \quad \text{where } \mathbf{M} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}. \quad (2.4)$$

The curl is directed perpendicular to the plane, and its magnitude is

$$\Omega \equiv \nabla \times \mathbf{F} \cdot \mathbf{e}_z = \partial_x F_y - \partial_y F_x = (\alpha - \gamma) \partial_{xy} U + \beta (\partial_{yy} U - \partial_{xx} U). \quad (2.5)$$

Usually this will not vanish, so these forces are curl forces. The familiar class of conservative forces, for which $\Omega = 0$, corresponds to $\alpha = \gamma$, $\beta = 0$.

The class (2.3) of Hamiltonian curl forces does not encompass the totality of curl forces. Inverting (2.3) to find $\nabla U(\mathbf{r})$, and then using $\nabla \times \nabla U(\mathbf{r}) = 0$, leads to the condition

$$\beta(\partial_x F_x - \partial_y F_y) - \alpha \partial_x F_y + \gamma \partial_y F_x = 0. \quad (2.6)$$

It is easy to find curl forces for which this cannot be satisfied for any choice of the constants α , β , γ . A simple example is $F_x = xy^2$, $F_y = x^3$, for which $\Omega = 3x^2 - 2xy$.

3. Special cases

(a) Linear curl forces

All such forces, namely

$$F_x = ax + by \quad \text{and} \quad F_y = cx + dy, \quad (3.1)$$

can be generated by Hamiltonians of the type (2.1). As is easily confirmed, the kinetic energy parameters and potential can be taken to be

$$\alpha = \frac{1}{c}, \quad \beta = 0, \quad \gamma = \frac{1}{b}, \quad U(\mathbf{r}) = -\frac{1}{2}acx^2 - bcxy - \frac{1}{2}bdy^2, \quad (3.2)$$

and the curl (2.5) is

$$\Omega = c - b. \quad (3.3)$$

The energy, defined as the value of the Hamiltonian, is a conserved function of positions and velocities that we denote by C_1 :

$$C_1 = \frac{1}{2}cv_x^2 + \frac{1}{2}bv_y^2 - \frac{1}{2}acx^2 - bcxy - \frac{1}{2}bdy^2. \quad (3.4)$$

These linear forces are integrable, so there is a second constant of motion. It can be written as

$$C_2 = -\frac{1}{2}c(d-a)v_x^2 + 2bcv_xv_y + \frac{1}{2}b(d-a)v_y^2 - \frac{1}{2}((-ac(d-a) + 2bc^2)x^2 + 2bc(d+a)xy + (bd(d-a) + 2b^2c)y^2). \quad (3.5)$$

For completeness, we write the evolution under a general linear curl force for arbitrary initial conditions. It is

$$\mathbf{r}(t) = (\mathbf{L}_+ \cdot \mathbf{r}(0) \cosh(t\sqrt{m_+}) + \mathbf{L}_+ \cdot \dot{\mathbf{r}}(0) \sinh(t\sqrt{m_+})\mathbf{R}_+ + (\mathbf{L}_- \cdot \mathbf{r}(0) \cosh(t\sqrt{m_-}) + \mathbf{L}_- \cdot \dot{\mathbf{r}}(0) \sinh(t\sqrt{m_-})\mathbf{R}_-, \quad (3.6)$$

in which $\mathbf{L}_{+,-}$ and $\mathbf{R}_{+,-}$ are the left and right eigenvectors of the force coefficient matrix, i.e.

$$\mathbf{m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{m} \cdot \mathbf{R}_{+,-} = m_{+,-}\mathbf{R}_{+,-}, \quad \mathbf{m}^T \cdot \mathbf{L}_{+,-} = m_{+,-}\mathbf{L}_{+,-}. \quad (3.7)$$

A physical example of such a linear curl force is that exerted on a polarizable particle by a monochromatic paraxial optical field, represented by a scalar wave function $\psi(\mathbf{r})$, near an optical vortex. The contribution from the imaginary part of the polarizability (representing dipole radiation) is proportional to [2]

$$\mathbf{F}(\mathbf{r}) = \text{Im}\psi^*(\mathbf{r})\nabla\psi(\mathbf{r}). \quad (3.8)$$

Near a vortex,

$$\psi(\mathbf{r}) = ux + vy, \quad (3.9)$$

where u and v are complex constants. The force—azimuthally directed—is

$$\mathbf{F}(\mathbf{r}) = \text{Im}[u^*v](-y, x) = \text{Im}[u^*v]r\mathbf{e}_\varphi, \quad (3.10)$$

corresponding to $a = d = 0$, $b = -c = -\text{Im}[u^*v]$ in (3.1), and curl $\Omega = 2\text{Im}[u^*v]$ in (3.3). The two conserved quantities (3.4) and (3.5) are

$$C_1 = \frac{1}{2}v_x^2 - \frac{1}{2}v_y^2 + \text{Im}[u^*v]xy \quad \text{and} \quad C_2 = v_xv_y - \frac{1}{2}\text{Im}[u^*v](x^2 + y^2). \quad (3.11)$$

(A note of caution. In optics, the term ‘curl force’ is sometimes used in a different sense [5,6]: to denote forces that can be represented as the curl of a vector (i.e. divergenceless forces), rather than, as here, forces whose curl is non-zero.)

(b) Azimuthal curl forces

In this section and the next, we use the ‘diagonal negative mass’ version of (2.1), for which $\alpha = -\gamma = 1$, $\beta = 0$. The Hamiltonian is

$$H(\mathbf{r}, \mathbf{p}) = \frac{1}{2}(p_x^2 - p_y^2) + U(\mathbf{r}), \quad (3.12)$$

and the force (2.3) is

$$F_x = \ddot{x} = -\partial_x U \quad \text{and} \quad F_y = \ddot{y} = +\partial_y U. \quad (3.13)$$

Now, take

$$U(\mathbf{r}) = f(xy). \quad (3.14)$$

The force (in general nonlinear) is

$$F_x = -yf'(xy) \quad \text{and} \quad F_y = +xf'(xy), \quad \text{i.e. } \mathbf{F}(\mathbf{r}) = rf'(\frac{1}{2}r^2 \sin(2\varphi))\mathbf{e}_\varphi. \quad (3.15)$$

This is azimuthally directed. The optical vortex curl force (3.10) is a special case, for which $f(xy) = xy$; for this case, the force has rotational symmetry, unlike the general case of (3.15), which depends on ϕ . The curl is

$$\Omega(\mathbf{r}) = \partial_x F_y - \partial_y F_x = 2(f'(xy) + xyf''(xy)). \quad (3.16)$$

The conserved energy is

$$C_1 = \frac{1}{2}(v_x^2 - v_y^2) + f(xy). \quad (3.17)$$

We cannot find an additional constant of motion, and conjecture (supported by numerical explorations) that, in general, this class of azimuthal forces is non-integrable (an exception is the linear case, for which there are the two conserved quantities (3.11)).

We remark that (3.15) is not the most general azimuthal force. For example, the previously studied rotationally symmetric forces $\mathbf{F}(\mathbf{r}) = F_\phi(r)\mathbf{e}_\phi$ [1] do not fall into this Hamiltonian class for general $F_\phi(r)$.

(c) Radial curl forces

Now, take the following form for $U(\mathbf{r})$ in (3.12):

$$U(\mathbf{r}) = f(\frac{1}{2}(x^2 - y^2)). \quad (3.18)$$

For this case, the force (3.13) is directed radially

$$\left. \begin{aligned} F_x &= -xf'(\frac{1}{2}(x^2 - y^2)), & F_y &= -yf'(\frac{1}{2}(x^2 - y^2)), \\ \text{i.e. } \mathbf{F}(\mathbf{r}) &= -rf'(\frac{1}{2}r^2 \cos(2\varphi))\mathbf{e}_r. \end{aligned} \right\} \quad (3.19)$$

The curl is

$$\Omega(\mathbf{r}) = -2xyf''(\frac{1}{2}(x^2 - y^2)). \quad (3.20)$$

Again, energy is conserved,

$$C_1 = \frac{1}{2}(v_x^2 - v_y^2) + f(\frac{1}{2}(x^2 - y^2)) = \text{const.} \quad (3.21)$$

Now, there is a second constant of motion, namely

$$C_2 = xv_y - yv_x = r^2\dot{\phi}. \quad (3.22)$$

Therefore, all these radial curl forces are integrable. Of course, C_2 is just the angular momentum, whose conservation is physically obvious, because for radial forces there is no torque about the

origin. Because of the ϕ dependence, the force (3.19) does not possess rotational invariance [1]. Therefore, the conventional association between conservation laws and symmetry, originating in Noether's theorem, does not always apply to curl forces.

As with the azimuthal Hamiltonian forces, the radial forces (3.19) constitute a special class because of the particular ϕ dependence. But, angular momentum is conserved for all radial forces, Hamiltonian or not.

(d) Shear curl forces

These are forces directed in one direction (x , say), but depending on both x and y , that is,

$$\mathbf{F}(\mathbf{r}) = F_x(x, y)\mathbf{e}_x. \quad (3.23)$$

The curl is

$$\boldsymbol{\Omega}(\mathbf{r}) = -\partial_y F_x(x, y). \quad (3.24)$$

The scheme of §2 enables two closely related special classes of shear forces to be created, from the Hamiltonians

$$H = \frac{1}{2}p_x^2 \pm p_x p_y \mp \frac{1}{2}p_y^2 + U(x \pm y). \quad (3.25)$$

From (2.3), or directly, these curl forces are

$$F_x(x, y) = -2U'(x \pm y). \quad (3.26)$$

Of course, for general shear forces, the motion is separable and therefore integrable, and the zero-force y motion is simply

$$y(t) = y(0) + v_y(0)t. \quad (3.27)$$

Therefore, the x motion can be generated by the Hamiltonian

$$H = \frac{1}{2}p_x^2 + U(x, y(0) + v_y(0)t), \quad U(x, y) = - \int_{\text{const}}^x dx' F(x', y). \quad (3.28)$$

The particular case $F(x, y) = x^2 + y^2$ is given in an intriguing paper [11]. However, the Hamiltonian (3.28) depends on time and on the initial conditions for $y(0)$ and $v_y(0)$. The first of these dependences can be eliminated by the choice

$$H = \frac{1}{2}p_x^2 + v_y(0)p_y + U(x, y), \quad (3.29)$$

giving (3.27) for $y(t)$ and the x curl force

$$F_x(\mathbf{r}) = \ddot{x} = -\partial_x U(x, y). \quad (3.30)$$

4. Curl forces, periodic orbits and closed loops

Curl forces imply severe restrictions on periodic orbits. To explain these, we first consider the general case of curl forces in three as well as two dimensions and not restricted to those originating in a Hamiltonian.

Imagine that the dynamics (1.1) generates a periodic orbit, that is, an orbit that is closed not only in position space but in the full position-velocity state space. Then, the velocity, and therefore the kinetic energy $\frac{1}{2}|v|^2$, must be unchanged after each traversal. However, Stokes's theorem implies that the change in kinetic energy around the orbit is, with s denoting arc length along the orbit,

$$\oint ds \frac{d}{ds} (\frac{1}{2}v \cdot v) = \oint d\mathbf{r} \cdot \ddot{\mathbf{r}} = \oint d\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) = \iint d\mathbf{S} \cdot \nabla \times \mathbf{F}(\mathbf{r}), \quad (4.1)$$

where the double integral is over any surface spanning the closed orbit. The only way to avoid contradiction is for the flux of $\nabla \times \mathbf{F}$ through the orbit to vanish.

There are three ways in which this can happen. Two of them occur when the orbit lies in a region of position space where $\nabla \times \mathbf{F} \cdot d\mathbf{S}$ would have the same sign for any spanning surface.

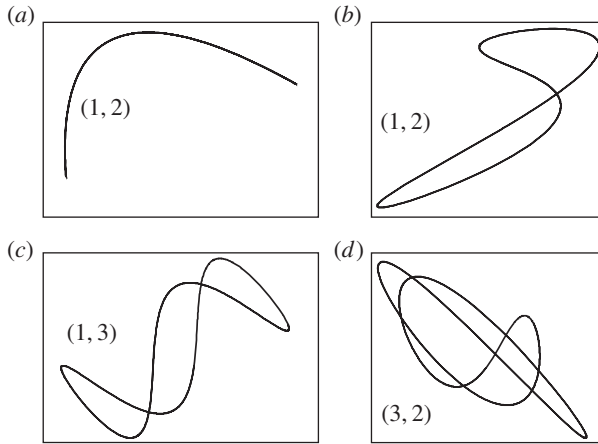


Figure 1. Periodic orbits generated by linear curl forces (3.1), calculated from (B7) for the following values of $A, D, x(0), y(0), v_x(0), v_y(0)$: (a) (1, 2, 1, 0, 0, 0); (b) (2, 0.2, 1, 1, 2, -1); (c) (1, 2, 1, 2, 1, 0); (d) (1, 3.5, 3, 4, 3, 4), with the indicated values of the commensurability parameters (M, N) .

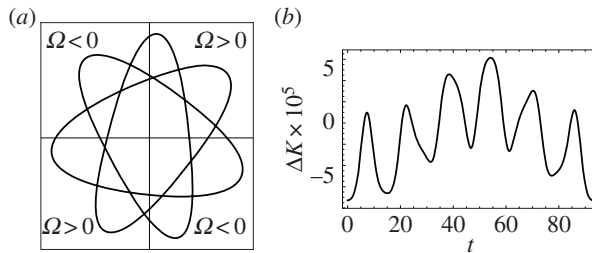


Figure 2. (a) A periodic orbit for the Hamiltonian (2.1) with $U(\mathbf{r}) = r^4/4$ and parameters $\alpha = 1, \beta = 0, \gamma = 0.9$, and initial conditions $x(0) = 0.1, y(0) = 0, v_x(0) = -0.005, v_y(0) = 0.04890$; (b) the energy change (4.2), for one period $0 \leq t < 93.235$ of the orbit in (a).

In the first, closed orbits are self-retracing (figure 1a), that is, enclosing no area. Such self-retracing periodic orbits correspond to particles released from rest, and are familiar in conventional Hamiltonian dynamics with time-reversal symmetry. In the second, the periodic orbits are self-crossing in a way that divides them into sections traversed in opposite senses (i.e. fluxes of opposite sign), with net flux zero (figure 1b–d). As explained in appendix B, both cases occur for linear curl forces in the plane (§3.1), where the curl is the same everywhere (equation (3.3)).

The third possibility is that the flux vanishes because the surface spanning a non-self-retracing and non-self-crossing orbit contains a line (or several lines) on which $\nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$, giving two (or more) positive and negative contributions to the surface integral, with the total flux cancelling. Figure 2a shows an example where $\Omega \sim xy$, which vanishes on the x - and y -axes. As a measure of the change ΔK in kinetic energy associated with the curl Ω , we subtract the part of the Hamiltonian (2.1) that generates a curl-free force, as follows:

$$\begin{aligned} \Delta K &\equiv H(\mathbf{r}, \mathbf{p}) - \left(\frac{1}{4}(\alpha + \gamma)(p_x^2 + p_y^2) + U(\mathbf{r})\right) \\ &= \beta p_x p_y + \frac{1}{4}(\alpha - \gamma)(p_x^2 - p_y^2). \end{aligned} \tag{4.2}$$

This quantity is shown in figure 2b for the periodic orbit in figure 2a. There are six intervals over which ΔK increases, and six for which ΔK decreases, corresponding to the $\Omega > 0$ and $\Omega < 0$ segments of the orbit in figure 2a.

Consider again general curl forces, but now not a periodic orbit but a closed curve $C(t) = \mathbf{r}(s, t)$ ($0 \leq s \leq 1$) at fixed time t in the state space \mathbf{r}, \mathbf{v} . Around this curve, the circulation is

$$J(t) = \oint_{C(t)} \mathbf{v} \cdot d\mathbf{r} = \int_0^1 ds \mathbf{v}(s, t) \cdot \partial_s \mathbf{r}(s, t), \quad (4.3)$$

where $0 \leq s < 1$ parametrizes points on the curve—i.e. s is a label that distinguishes different trajectories at time t . As time changes, $C(t)$ moves, because each point on it evolves. The rate of change of J is

$$\begin{aligned} \partial_t J(t) &= \int_0^1 ds (\partial_t \mathbf{v}(s, t) \cdot \partial_s \mathbf{r}(s, t) + \mathbf{v}(s, t) \cdot \partial_{st} \mathbf{r}(s, t)) \\ &= \int_0^1 ds (\partial_t \mathbf{v}(s, t) \cdot \partial_s \mathbf{r}(s, t) - \partial_s \mathbf{v}(s, t) \cdot \partial_t \mathbf{r}(s, t)) \\ &= \int_0^1 ds (\mathbf{F}(\mathbf{r}(s, t)) \cdot \partial_s \mathbf{r}(s, t) - \partial_s \mathbf{v}(s, t) \cdot \mathbf{v}(s, t)) \\ &= \int_0^1 ds (\mathbf{F}(\mathbf{r}(s, t)) \cdot \partial_s \mathbf{r}(s, t) - \frac{1}{2} \frac{d}{ds} v^2(s, t)) \\ &= \int_0^1 ds \mathbf{F}(\mathbf{r}(s, t)) \cdot \partial_s \mathbf{r}(s, t) = \oint_{C(t)} \mathbf{F} \cdot d\mathbf{r} = \iint_{\partial S(t)=C(t)} \nabla \times \mathbf{F} \cdot \mathbf{n} dS, \end{aligned} \quad (4.4)$$

where integration by parts has been used in the second line, and where $S(t)$ is any surface bounded by $C(t)$ with unit normal \mathbf{n} . Thus, J is not conserved for curl forces; moreover, $\partial_t J$ is the virtual work done by \mathbf{F} round C at time t .

This argument applies to any curl force. For the special class of curl forces which can be obtained from a Hamiltonian, that we have introduced here, we can also consider closed curves $C_p(t)$ in the phase space \mathbf{r}, \mathbf{p} , and the different circulation

$$J_p(t) = \oint_{C_p(t)} \mathbf{p} \cdot d\mathbf{r}. \quad (4.5)$$

The familiar argument for the conservation of Poincaré invariants in Hamiltonian dynamics [12] shows that this circulation is conserved.

5. Remarks and open questions

As we have shown, some members of the class of curl forces—by which we mean forces that are not the gradient of a potential and so are non-conservative, but also non-dissipative—can be generated by Hamiltonians. We conclude with some remarks and unanswered questions.

1. In appendix A, we explain why Hamiltonians that can generate velocity-independent forces, and are representable as power series in the momenta, must lie in the quadratic-anisotropic-kinetic class (2.1). This class generates curl forces of the restricted type (2.4). We have not excluded the possibility that more general velocity-independent forces could be generated by Hamiltonians that are non-analytic in the momenta, but this seems unlikely. A proof would be desirable. (To avoid misunderstanding, we refer to Hamiltonians with the same number of freedoms as the ambient space, i.e. without embedding the dynamics in a higher-dimensional space.)
2. If, as seems probable, there exist curl forces that cannot be generated from any Hamiltonian, a natural question is: what features of the dynamics would distinguish non-Hamiltonian curl forces from Hamiltonian ones? This amounts to asking which aspects of the symplectic structure enjoyed by Hamiltonian systems would be destroyed by curl forces of non-Hamiltonian type. In §4, we established some geometrical restrictions on periodic orbits, and the non-conservation of the natural generalization of the Poincaré loop integral.

A different distinction, pointed out to us by V. Gelfreich (2014, private communication), is the possible violation of the Hamiltonian property that the Lyapunov exponents of periodic orbits fall into pairs with equal magnitude and opposite sign. (Of course, the non-dissipative nature of curl dynamics means that all Lyapunov exponents, defined in position–velocity space rather than in phase space, must sum to zero).

- Now, we ask whether all position-dependent forces (1.1) can be represented by expressions of the optical type (3.8), for a suitable choice of the complex scalar wave function $\psi(x, y)$. In other words, can any curl force be realized as an optical force on a particle? The answer is no, notwithstanding the fact that $\psi(x, y) = \text{Re}[\psi(x, y)] + i \text{Im}[\psi(x, y)]$ implies the availability of two scalar functions: just what is needed to specify the force $F(x, y) = (F_x(x, y), F_y(x, y))$. The particular form of (3.8) imposes constraints on the forces that can be so represented, as described in appendix C.

It is also the case that not all optical forces of the type (3.8) can be generated from the Hamiltonian (2.1). The argument is based on the condition (2.6), with F given by (3.8), which must be satisfied by the complex scalar function $\psi(x, y)$ everywhere. This cannot be accomplished by adjusting the values of the three constants α, β, γ .

- We make some remarks about the applicability of Noether's theorem. For curl forces that cannot be derived from a Hamiltonian, the theorem is irrelevant because it refers to symmetries of the Hamiltonian (or Lagrangian), not the force. But, for the Hamiltonian curl forces considered in this paper, we need to be a little more careful. For, example, all the Hamiltonian radial curl forces considered in §3c possess the conserved angular momentum (3.22), but the force does not have rotation symmetry, and nor does the Hamiltonian (i.e. energy; (3.21)), so again Noether's theorem cannot be applied. (Of course, all our Hamiltonians are conserved, but this simply reflects the time translation invariance of the dynamics.) A remaining question is: in cases where there is a second constant of motion, for example (3.5) and (3.11), what physical symmetry does this represent? A referee raised the interesting question of whether there is an intuitive physical interpretation of the conserved quantities (3.11) for optical curl forces, which originate in Maxwell's equations and the Coulomb–Lorentz force; we do not know.
- Hamiltonian curl forces can be quantized by standard methods (e.g. the replacement $p = -i\hbar\nabla$ and suitable symmetrization) though if there are several conserved quantities the choice of which one to quantize requires care [13]. A further natural question is: how does the quantum mechanics of Hamiltonian systems with a curl force differ from the familiar quantum mechanics of systems with a conservative force?

Data accessibility. This research involves no data.

Acknowledgements. We thank Professor Vassily Gelfreich and Dr Jonathan Robbins for helpful comments and suggestions. M.V.B. thanks the Liu Bie Ju Centre for Mathematical Sciences of the City University of Hong Kong for generous hospitality while writing the first draft of this paper.

Authors' contributions. Both authors equally contributed.

Funding statement. Neither author received funding to carry out this research.

Competing interests. None.

Appendix A. Hamiltonian forces independent of velocity

We can formulate the condition for the force to be independent of velocity in the following way. Considering the two-dimensional case, we start from the velocity

$$v_i = \partial_{p_i} H(\mathbf{r}, \mathbf{p}), \quad (i = 1, 2), \quad (\text{A } 1)$$

from which the force is (using the summation convention)

$$F_i(\mathbf{r}, \mathbf{p}) = \ddot{r}_i = \dot{v}_i = v_j \partial_{p_j} H + \dot{p}_j \partial_{p_j} H = \partial_{p_j} H \partial_{p_j} H - \partial_{p_j} H \partial_{r_j} H. \quad (\text{A } 2)$$

We assume (A 1) can be inverted, to give, at least locally, the momentum as a function of position and velocity, which we write as

$$\mathbf{p} = \Pi(\mathbf{r}, \mathbf{v}). \quad (\text{A } 3)$$

The desired condition, that the force be independent of velocity, is

$$\partial_{v_i} F_j(\mathbf{r}, \Pi(\mathbf{r}, \mathbf{v})) = 0 \quad \text{for all } i, j. \quad (\text{A } 4)$$

Now,

$$\partial_{v_i} F_j(\mathbf{r}, \Pi(\mathbf{r}, \mathbf{v})) = \partial_{v_i} \Pi_k(\mathbf{r}, \mathbf{v}) \partial_{p_k} F_j(\mathbf{r}, \Pi(\mathbf{r}, \mathbf{v})), \quad (\text{A } 5)$$

and, by differentiating (A 1), with respect to v_j , we have

$$\delta_{ij} = \partial_{p_i p_k} H(\mathbf{r}, \mathbf{p}) \partial_{v_j} \Pi_k. \quad (\text{A } 6)$$

The solution is

$$\partial_{v_i} \Pi_k(\mathbf{r}, \dot{\mathbf{r}}) = \left[(\partial_{p_i p_j} H)^{-1} \right]_{ik}, \quad (\text{A } 7)$$

provided the Hessian matrix of momentum derivatives can be inverted. Thus the condition (A 4) becomes, using (A 5),

$$\left[(\partial_{p_i p_j} H)^{-1} \right]_{ik} \partial_{p_k} F_j(\mathbf{r}, \Pi(\mathbf{r}, \dot{\mathbf{r}})) = 0 \quad \text{for all } i, j. \quad (\text{A } 8)$$

This is simply a matrix product, so the condition of velocity independence amounts to the requirement that the phase-space force function (A 2) is independent of momentum. This is certainly the case for the Hamiltonian (2.1), as (2.4) demonstrates.

The argument that such Hamiltonians must fall in the class (2.1) turned out to be unexpectedly subtle (at least we could not find a simpler proof). Therefore, we present the argument in its essence, by first considering the one-dimensional case, i.e. Hamiltonians of the form $H(x, p)$, for which the phase-space force is (cf. (A 2))

$$F(x, p) = \partial_p H(x, p) \partial_{p,x} H(x, p) - \partial_x H(x, p) \partial_{p,p} H(x, p). \quad (\text{A } 9)$$

We consider the fairly general class of Hamiltonians that are power series in p of degree N , that is,

$$H_N(x, p) = \sum_{n=0}^N f_n(x) p^n, \quad (\text{A } 10)$$

in which we assume $f_N(x) \neq 0$, and seek the condition that the force (A 9) is independent of p . The force is a power series of degree $2N-2$, and we require that the coefficients of all terms p^n must vanish, except that of the constant term p^0 . We denote differentiation by x by primes.

The case $N = 1$ is easy

$$H_1(x, p) = f_0(x) + f_1(x)p \Rightarrow F_1 = f_1(x)f_1'(x). \quad (\text{A } 11)$$

For $N = 2$,

$$\begin{aligned} H_2(x, p) &= f_0(x) + f_1(x)p + f_2(x)p^2 \\ &\Rightarrow F_2 = f_1(x)f_1'(x) - 2f_2f_0'(x), \end{aligned} \quad (\text{A } 12)$$

in which we have replaced $f_2(x)$ by a constant in order to eliminate terms involving p and p^2 in F_2 . This case is just the one-dimensional counterpart of (2.1), including the linear term in p which was omitted because it can be eliminated by a linear canonical transformation.

For $N > 2$, we proceed by determining the conditions for the vanishing of all coefficients of powers of p successively, starting with p^{2N-2} . From (A 9) (and now omitting the explicit x dependences), this highest term and the condition for its vanishing are

$$Nf_N f'_N p^{2N-2} \Rightarrow f'_N = 0. \quad (\text{A } 13)$$

By inspection, this also guarantees the vanishing of the term in p^{2n-3} . For the vanishing of the lower terms, we find (again by inspection) the conditions

$$p^{2N-4} : f'_{N-2} = \frac{(N-1)f_{N-1}f'_{N-1}}{Nf_N}, \quad p^{2N-5} : f'_{N-3} = \frac{(N-2)f_{N-2}f'_{N-2}}{(N-1)f_{N-1}} \dots, \quad (\text{A } 14)$$

enabling all derivatives $f'_0, f'_1, \dots, f'_{N-2}$ to be expressed in terms of f'_{N-1}

$$f'_{N-m} = \frac{(N-m+1)f_{N-m+1}f'_{N-1}}{Nf_N} \equiv C_N(N-m+1)f_{N-m+1},$$

where
$$C_N = \frac{f'_{N-1}}{Nf_N}. \quad (\text{A } 15)$$

Thus

$$f'_n = C_N(n+1)f_{n+1}. \quad (\text{A } 16)$$

For the derivatives in the force (A 9), we now have

$$\left. \begin{aligned} \partial_p H &= \sum_{n=1}^N n f_n p^{n-1} = \sum_{n=0}^{N-1} (n+1) f_{n+1} p^n, \\ \partial_x H &= \sum_{n=0}^{N-1} f'_n p^n = C_N \sum_{n=0}^{N-1} (n+1) f_{n+1} p^n, \\ \partial_{x,p} H &= C_N \sum_{n=1}^{N-1} n(n+1) f_{n+1} p^{n-1} = C_N \sum_{n=0}^{N-2} (n+1)(n+2) f_{n+2} p^n, \\ \partial_{p,p} H &= \sum_{n=2}^N n(n-1) f_n p^{n-2} = \sum_{n=0}^{N-2} (n+2)(n+1) f_{n+2} p^n, \end{aligned} \right\} \quad (\text{A } 17)$$

showing immediately that the force (A 9) vanishes. This unexpected result indicates that the vanishing of the highest force coefficients $p^{N-2}, p^{N-1}, p^N \dots p^{2N-2}$ ensures the vanishing of all the lower force coefficients, including the coefficient of p^0 , that is, the force itself. Therefore, the only Hamiltonian of the type (A 10) for which the condition of velocity independence gives a non-zero force is the case $N = 2$, as claimed.

This is a one-dimensional argument. The two-dimensional case is similar but more intricate. We give just the outline. The generalization of the power series (A 10) is

$$H_N(\mathbf{r}, \mathbf{p}) = \sum_{n=0}^N \left(\sum_{j=0}^n f_{n,j}(\mathbf{r}) p_1^j p_2^{n-j} \right). \quad (\text{A } 18)$$

In the force (A 2), the highest terms are again of order $|p|^{2N-2}$, and involve the $N+1$ coefficients $f_{N,j}$ and their derivatives with respect to x_1 and x_2 . There are $2N+2$ such derivatives. In the force component F_x the coefficients of $p_1^{2N-2}, p_1^{2N-3} p_2, \dots, p_1 p_2^{2N-3}, p_2^{2N-2}$, that is, $2N-1$ coefficients, must all vanish, and similarly for F_y . Therefore, we have $4N-2$ linear equations, which exceeds the number of derivatives $2N+2$ if $N > 2$. Therefore, all derivatives $\partial_x f_{N,j}(\mathbf{r})$ and $\partial_y f_{N,j}(\mathbf{r})$ must vanish, in analogy with (A 13) in the one-dimensional case.

The rest of the argument proceeds similarly, with the result that the only non-zero velocity-independent force must be quadratic, as in the case that we have studied (after eliminating terms linear in p), i.e. (2.1).

Appendix B. Periodic orbits for linear curl forces must be self-retracing or self-crossing

For linear curl forces, we can write the equation of motion (3.1) as

$$\ddot{\mathbf{r}}(t) = \mathbf{m}\mathbf{r}(t), \quad \text{where } \mathbf{m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (\text{B } 1)$$

in which $b \neq c$ for curl forces (cf. (3.3)). The general solutions (3.6) involve $\cosh(t\sqrt{m_{\pm}})$ and $\sinh(t\sqrt{m_{\pm}})$, where m_{\pm} are the eigenvalues of \mathbf{m} , namely

$$m_{\pm} = \frac{1}{2} \left(a + d \pm \sqrt{4bc + (a - d)^2} \right). \quad (\text{B } 2)$$

To represent oscillations, the arguments of \cosh and \sinh must be imaginary, so m_{\pm} must both be negative, and to be periodic these arguments must also be commensurable.

For commensurability with exponent ratio M/N , where M, N are integers with no common factor, we require

$$M^2 m_+ = N^2 m_-, \quad (\text{B } 3)$$

which after some algebra implies

$$\left. \begin{aligned} bc &= \frac{ad(M^4 + N^4) - (a^2 + d^2)M^2 N^2}{(M^2 + N^2)^2}, \\ m_+ &= N^2 \frac{a+d}{M^2 + N^2}, \quad m_- = M^2 \frac{a+d}{M^2 + N^2}. \end{aligned} \right\} \quad (\text{B } 4)$$

For oscillation, i.e. negative eigenvalues, we require $a + d < 0$. Without loss of essential generality, we can take $b = 1$, and therefore write, in the dynamical matrix \mathbf{m} , the coefficients generating periodic orbits:

$$\left. \begin{aligned} a &= -A, \quad b = 1, \\ c &= \frac{AD(M^4 + N^4) - (A^2 + D^2)M^2 N^2}{(M^2 + N^2)^2}, \quad d = -D \end{aligned} \right\} \quad (\text{B } 5)$$

with

$$A + D > 0.$$

It is now easy to solve (B 1) to get the orbits in terms of the initial positions and velocities. Defining

$$\gamma \equiv \sqrt{\frac{A + D}{M^2 + N^2}}, \quad (\text{B } 6)$$

we find the following explicit version of (3.6):

$$\left. \begin{aligned} x(t) &= x(0) \frac{(AM^2 - DN^2) \cos \gamma Mt + (DM^2 - AN^2) \cos \gamma Nt}{(A+D)(M^2 - N^2)} \\ &\quad + y(0) \frac{\cos \gamma Nt - \cos \gamma Mt}{\gamma^2 (M^2 - N^2)} \\ &\quad + v_x(0) \frac{(AM^2 - DN^2)N \sin \gamma Mt + (DM^2 - AN^2)M \sin \gamma Nt}{\gamma(A+D)MN(M^2 - N^2)} \\ &\quad + v_y(0) \frac{M \sin \gamma Nt - N \sin \gamma Mt}{\gamma^3 MN(M^2 - N^2)}, \\ y(t) &= x(0) \frac{(DM^2 - AN^2)(AM^2 - DN^2)(\cos \gamma Nt - \cos \gamma Mt)}{\gamma^2 (M^2 - N^2)} \\ &\quad + y(0) \frac{(AM^2 - DN^2) \cos \gamma Nt + (DM^2 - AN^2) \cos \gamma Mt}{(A+D)(M^2 - N^2)} \\ &\quad + v_x(0) \frac{(DM^2 - AN^2)(AM^2 - DN^2)(M \sin \gamma Nt - N \sin \gamma Mt)}{\gamma^3 MN(M^2 - N^2)} \\ &\quad + v_y(0) \frac{(AM^2 - DN^2)M \sin \gamma Nt + (DM^2 - AN^2)N \sin \gamma Mt}{\gamma(A+D)MN(M^2 - N^2)}. \end{aligned} \right\} \quad (\text{B } 7)$$

The period of the orbit is $T = 2\pi/\gamma$. If the particle starts from rest (i.e. $v(0) = 0$), it follows from (B 7) that the only trigonometric functions in $\mathbf{r}(t)$ are $\cos\gamma Mt$ and $\cos\gamma Nt$, so $\mathbf{r}(t) = \mathbf{r}(T - t)$. Therefore, the orbit from $t = 0$ to $T/2$ is retraced from $t = T/2$ to T (figure 1a). If the particle does not start from rest, the occurrence in (B 7) of $\sin\gamma Mt$ and $\sin\gamma Nt$ means that the orbit is not self-retracing; instead, it crosses itself at pairs of times t_i, t_j , i.e. $\mathbf{r}(t_i) = \mathbf{r}(t_j)$ (figure 1b,c,d). These times depend not only on M and N but also on A and D and the initial conditions $\mathbf{r}(0)$ and $v(0)$.

This argument fails for the simplest commensurability, i.e. $M = N = 1$, for which the eigenvalues are degenerate. Then the general solution of (B 1) involves not only the trigonometric functions but also terms in which these are multiplied by t , a familiar situation for degenerate non-symmetric matrices such as (B 1) [14]. These terms must vanish for periodic orbits, which therefore represent a special class of solutions, in which the initial conditions $v_x(0)$ and $v_y(0)$ cannot be freely specified. Careful analysis of (B 7) gives these solutions as the straight lines

$$x(t) = \frac{2y(t)}{A - D} = x(0) \cos\left(t\sqrt{\frac{1}{2}(A + D)}\right) + v_x(0) \frac{\sin\left(t\sqrt{\frac{1}{2}(A + D)}\right)}{\sqrt{\frac{1}{2}(A + D)}}, \quad (\text{B } 8)$$

which are both periodic and self-retracing, consistent with the argument in §4.

Appendix C. Optical forces cannot reproduce all curl forces

An easy consequence of (3.8) is that these optical forces satisfy

$$\mathbf{F} \cdot \nabla \times \mathbf{F} = 0, \quad (\text{C } 1)$$

implying that \mathbf{F} is everywhere normal to a foliation of three-dimensional space (optical wavefront surfaces $\arg[\psi] = \text{constant}$). This restricts the possible form of optical forces in space. In two dimensions, it is automatically satisfied: if $\mathbf{F} = (F_x(x, y), F_y(x, y))$, then $\nabla \times \mathbf{F}$ lies in the z -direction. But, even in two dimensions, all curl forces cannot be reproduced from (3.8) by any wave function $\psi(x, y)$. It suffices to show this for linear forces (3.1). The possible functions $\psi(x, y)$ can be at most quadratic in x and y , that is, they must take the form

$$\psi(x, y) = c_0 + c_1x + c_2y + \frac{1}{2}c_{11}x^2 + c_{12}xy + \frac{1}{2}c_{22}y^2, \quad (\text{C } 2)$$

in which without essential loss of generality we can take c_0 real. This generates optical forces including not only the desired linear terms in (3.1) but also cubic, quadratic and constant terms, whose coefficients must vanish. Applying these conditions to direct calculation of \mathbf{F} from (3.8) shows that the off-diagonal coefficients b and c in (3.1) are restricted as follows. If $c_1 = c_2 = 0$ and $c_0 \neq 0$, then $b = c$; this is the case where \mathbf{F} is a potential force, i.e. not a curl force. If $c_0 = 0$ and $c_1 \neq 0$ and $c_2 \neq 0$, $a = d = 0$ and $b = -c$; this corresponds to the azimuthal linear curl force (3.10). This exhausts the possible cases (if $c_0 \neq 0$ and either c_1 or c_2 is non-zero, application of the conditions leads to $\mathbf{F} = 0$).

References

1. Berry MV, Shukla P. 2012 Classical dynamics with curl forces, and motion driven by time-dependent flux. *J. Phys. A* **45**, 305201. (doi:10.1088/1751-8113/45/30/305201)
2. Berry MV, Shukla P. 2013 Physical curl forces: dipole dynamics near optical vortices. *J. Phys. A* **46**, 422001. (doi:10.1088/1751-8113/46/42/422001)
3. Chaumet PC, Nieto-Vesperinas M. 2000 Time-averaged total force on a dipolar sphere in an electromagnetic field. *Opt. Lett.* **25**, 1065–1067. (doi:10.1364/OL.25.001065)
4. Shimizu Y, Sasada H. 1998 Mechanical force in laser cooling and trapping *Am. J. Phys.* **66**, 960–967. (doi:10.1119/1.19006)
5. Albaladejo S, Marqués MI, Laroche M, Sáenz JJ. 2009 Scattering forces from the curl of the spin angular momentum. *Phys. Rev. Lett.* **102**, 113602. (doi:10.1103/PhysRevLett.102.113602)

6. Gómez-Medina R, Nieto-Vesperinas M, Sáenz JJ. 2011 Nonconservative electric and magnetic optical forces on submicron dielectric particles. *Phys. Rev. A* **83**, 033825. (doi:10.1103/PhysRevA.83.033825)
7. Elishakoff I. 2005 Controversy associated with the so-called 'follower forces': critical overview. *Appl. Math. Rev.* **58**, 117–142. (doi:10.1115/1.1849170)
8. Roberts JAG, Quispel GRW. 1992 Chaos and time-reversal symmetry. Order and chaos in reversible dynamical systems. *Phys. Rep.* **216**, 63–177. (doi:10.1016/0370-1573(92)90163-T)
9. Gutzwiller MC. 1973 The anisotropic Kepler problem in two dimensions. *J. Math. Phys.* **14**, 139–152. (doi:10.1063/1.1666164)
10. Gutzwiller MC. 1990 *Chaos in classical and quantum mechanics*. New York, NY: Springer.
11. Nucci MC, Leach PGL. 2011 Some Lagrangians for systems without a Lagrangian. *Phys. Scr.* **83**, 035007. (doi:10.1088/0031-8949/83/03/035007)
12. Arnold VI. 1978 *Mathematical methods of classical mechanics*. New York, NY: Springer.
13. Nucci MC. 2012 From Lagrangian to quantum mechanics with symmetries. *J. Phys. Conf. Ser.* **380**, 012008. (doi:10.1088/1742-6596/380/1/012008)
14. Berry MV. 2004 Physics of nonhermitian degeneracies. *Czech. J. Phys.* **54**, 1040–1047.