IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor **47** (2014) 315203 (14pp) doi:10.1088/1751-8113/47/31/315203

Superoscillations and supershifts in phase space: Wigner and Husimi function interpretations

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Received 1 June 2014
Accepted for publication 13 June 2014
Published 18 July 2014

Abstract

Superoscillations, namely regions where a band-limited function $f(x)$ varies faster than the fastest of its Fourier components k, generate the illusion that the Fourier content is 'supershifted' so as to lie outside the spectrum of the function. The relation between supershifts and superoscillations, central to the quantum weak measurements scheme, is explored in terms of two different representations of the local Fourier transform in the 'phase space' (x, k). The Wigner function $W(x, k)$, regarded as a function of k for fixed x, inherits the band-limited property of $f(x)$. Neverthless, its local k average can lie outside the spectrum because W, although real, posesses negative values. The local Wigner average of k equals the local wavenumber at x (local weak value of momentum), defined as the phase variation $k_{\mathrm{loc}}(x) = \partial_x \arg f(x)$. By contrast, the Husimi function $H(x, k)$, i.e. the windowed Fourier transform with window width L, corresponding to squeezing of the coherent state associated with (x, k) (and representing the pointer wavefunction after a weak measurement), is positive-definite. But it is not band-limited, and the local Husimi average of k equals k_{loc} if L is small enough. These properties are illustrated numerically with two superoscillatory functions.

Keywords: phase space, weak measurement, wavenumber
PACS numbers: 02.30.Nw, 02.30.Uu, 03.65.Ca, 03.65.Ta, 03.65.Vf, 42.25.Hz

1. Introduction

Superoscillation refers to a property of a band-limited function $f(x)$: in regions where $f(x)$ is very small, it can oscillate faster than its fastest Fourier component [1–5]. Such functions may be non-periodic or periodic. When written in the form

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-k_{\max}}^{k_{\max}} \mathrm{d}k \bar{f}(k) \exp(\mathrm{i}kx) \,(\text{non-periodic}) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{-k_{\max}}^{k_{\max}} \bar{f}_k \exp(\mathrm{i}kx) \,(2\pi\text{-periodic}, \; k \text{ integer}), \end{aligned} \tag{1.1}$$

superoscillation means that there are regions of x where $f(x)$ varies faster than $\exp(\pm \mathrm{i}k_{\max} x)$. One way to quantify this is by the local rate of change of the phase of $f(x)$; this is the local wavenumber

$$\begin{aligned} k_{\mathrm{loc}}(x) &= \partial_x \arg f(x) \\ &= \partial_x \operatorname{Im}[\log(f(x))] \\ &= \operatorname{Im}\left[\frac{\partial_x f(x)}{f(x)}\right] \\ &= \frac{\operatorname{Im}\left[f^*(x)\, \partial_x f(x)\right]}{|f(x)|^2}. \end{aligned} \tag{1.2}$$

In terms of $k_{\mathrm{loc}}(x)$, superoscillation means $|k_{\mathrm{loc}}(x)| > k_{\max}$, giving rise to the illusion that $f(x)$ possesses wavenumbers k ouside the domain $|k| < k_{\max}$ of $\bar{f}(k)$. It is as though the Fourier content is 'supershifted' outside the spectrum. Superoscillations have been extensively studied: see for example the review [1]. Our purpose here is to shed further light on the illusion by relating the supershift to several interpretations of the concept of local Fourier transform.

A physical context in which the notions of superoscillation and supershift originated is due to Aharonov [6–10]. In this framework, the supershift can be interpreted as an unexpectedly large displacement of a pointer in a 'quantum weak measurement', in which the choice of the position x in a superoscillatory region is an example of 'post-selection' and $k_{\mathrm{loc}}(x)$ is the local weak value of momentum [11]. These physical aspects have been considered extensively and we do not discuss them here.

We choose to regard x as position, and k (continuous or discrete as in (1)) as the conjugate momentum, with (x, k) constituting phase space. In section 2 we connect superoscillations to the Wigner phase-space distribution function $W(x, k)$, by explaining that $k_{\mathrm{loc}}(x)$ can be interpreted as the mean value of k for fixed x. There is an apparently paradoxical aspect of this, because if $f(x)$ is band-limited then $W(x, k)$ is band-limited too. The reason $W(x, k)$ can generate mean values outside its k spectrum is that this function, although real and normalized in the sense that its phase-space integral is zero, can be negative for some regions of x. This makes sense, because the negative values of W reflect interference oscillations in the function $f(x)$ [12], and superoscillations are a result of near-perfect destructive interference.

In section 3 we interpret the supershift differently, by relating $k_{\mathrm{loc}}(x)$ to the windowed Fourier transform, that is, the Fourier transform not of $f(x)$ itself but of $f(x)$ modified by selecting a region whose width is specified by a distance L. For Gaussian windows, the square of this windowed transform is the Husimi phase-space distribution function $H(x, k)$. The Husimi function is also is the intensity of the overlap of $f(x)$ with a coherent state; in this interpretation, L parameterizes the squeezing of the state. As will be explained, the Husimi function gives the pointer wavefunction after a weak measurement.

Unlike $W(x, k)$, $H(x, k)$ is positive semidefinite, so for fixed x it can be regarded as a probability distribution in k. But windowing has destroyed the band-limited property of $f(x)$,

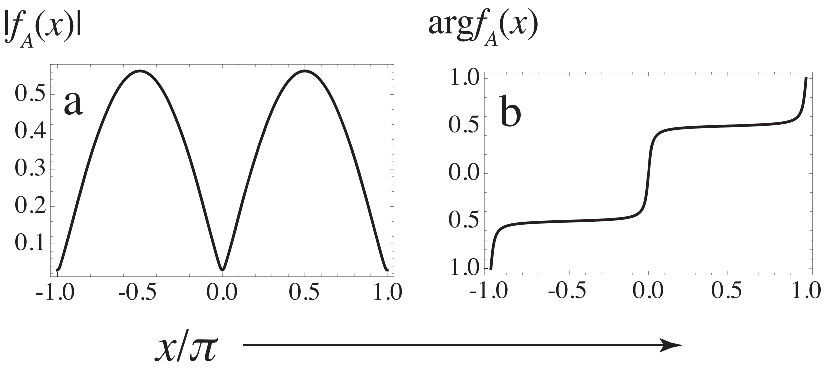

Figure 1. Modulus and phase of $f_A(x)$ for $a = 0.9$, showing superoscillations near $x = 0$ mod π.

so the mean k can lie outside the spectral range $|k| < k_{\max}$. And indeed we will see that for x in the superoscillatory region and sufficiently narrow window width L the Husimi mean momentum becomes equal to $k_{\text{loc}}(x)$, corresponding, in the physical interpretation, to the supershift of a pointer in a weak measurement.

It will sometimes be convenient to deal with functions normalized as follows:

$$\begin{array}{l} \int_{-\infty}^{\infty} \mathrm{d}x \, \left| f(x) \right|^2 = \int_{-\infty}^{\infty} \mathrm{d}\bar{k} \, \left| \bar{f}(k) \right|^2 = 1 \text{ (non-periodic)} \\ \int_{-\pi}^{\pi} \mathrm{d}x \, \left| f(x) \right|^2 = \sum_{-k_{\max}}^{k_{\max}} \left| \bar{f}_k \right|^2 = 1 \text{ (}2\pi\text{-periodic)} \end{array} . \tag{1.3}$$

To illustrate the general theory we will use two particular functions $f_A(x)$ and $f_B(x)$. The first [11] is extremely simple:

$$f_A(x) = \frac{\exp(\mathrm{i}x) - a \exp(-\mathrm{i}x)}{\sqrt{2\pi \left(1 + a^2\right)}}. \tag{1.4}$$

This is 2π-periodic, and obviously band-limited because it contains only two momenta: $k = 1$ and $k = -1$. The small value of the modulus $|f_A(x)|$ and rapid variation of the phase $\arg f_A(x)$, when a is close to unity and x is near zero (mod π), correspond to the rudimentary superoscillations of $f_A(x)$, as illustrated in figure 1. (For this simple function, superoscillations are not evident in $\mathrm{Re} f_A$ or $\mathrm{Im} f_A$.)

The phase and the local wavenumber (1.2) are

$$\begin{aligned} \arg f_A(x) &= \arctan\left(\frac{1 + a}{1 - a} \tan x \right), \\ k_{\text{loc}A}(x) &= \frac{1 - a^2}{1 + a^2 - 2a \cos 2x} \end{aligned} \tag{1.5}$$

It is interesting to note that the mean value of $k_{\text{loc}A}(x)$ over $(0 \leqslant x < \pi)$ is $+1$ if $a < 1$ and -1 if $a > 1$, corresponding to the dominant wavenumber in $f_A(x)$. As figure 2 shows, for a close to unity, $k_{\text{loc}} < 1$—that is, the local wavenumber lies within the spectral range $|k| \leqslant 1$–unless x is close to 0 (mod π), where the limiting behaviour is

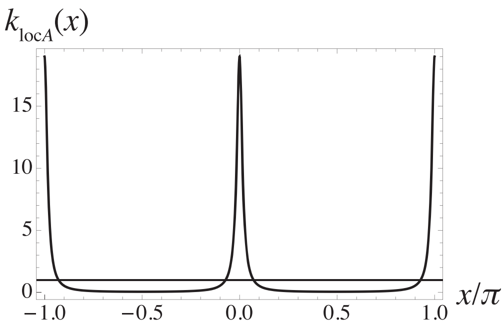

Figure 2. Local wavenumber (1.5) for $f_A(x)$, for $a = 0.9$; the superoscillatory region is $k_{\mathrm{loc}A}(x) > 1$.

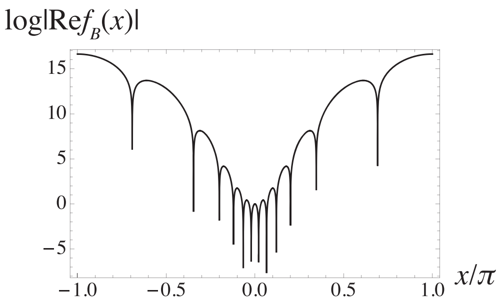

Figure 3. Superoscillatory function $f_B(x)$ (equation (1.7)) for $a = 4$, $N = 6$.

$$f_A(x) \approx \sqrt{\frac{\varepsilon^2 + 4x^2}{4\pi}} \exp\left(\frac{2\mathrm{i}x}{\varepsilon}\right), \quad k_{\mathrm{loc}A} \approx \frac{2}{\varepsilon} \quad (a = 1 - \varepsilon, x \approx 0). \qquad (1.6)$$

The exact maximum value of the local wavenumber, corresponding to superoscillation, is $k_{\mathrm{loc}A}(0) = (1 + a)/(1 - a)$ (= 19 for the example in figure 2), and the corresponding value of f at the origin is $\varepsilon/2\sqrt{\pi}$ (= 0.254 in figure 1).

The second function, variants of which have been studied several times before [5, 13], exhibits strong superoscillations:

$$f_B(x) = C\left(\cos\frac{1}{2}x + \mathrm{i}a \sin\frac{1}{2}x\right)^{2N} \quad (a > 1, N \gg 1) \qquad (1.7)$$

(here and hereafter, C will denote an unspecified normalization constant). We are interested in a real and N large, but remark that $f_B(x)$ can reproduce $f_A(x)$ as a special case if a is imaginary and $N = 1/2$.

The 2π-periodic function $f_B(x)$ is depicted in figure 3. The rapid variations near $x = 0$ correspond to superoscillations, where the local wavenumber (1.2) is

$$k_{\mathrm{loc}B}(x) = \frac{aN}{\cos^2\left(\frac{1}{2}x\right) + a^2 \sin^2\left(\frac{1}{2}x\right)} \to aN \text{ as } x \to 0 \qquad (1.8)$$

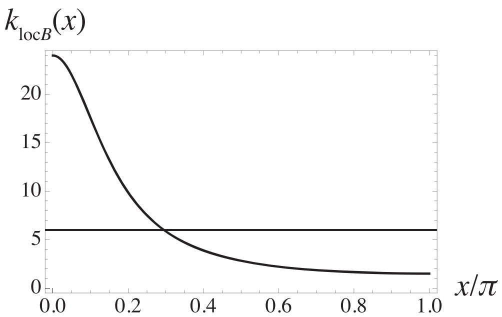

Figure 4. Local wavenumber (1.8) for $f_B(x)$, for $a=4$, $N=6$; the superoscillatory region is $k_{\mathrm{loc}}(x)>6$.

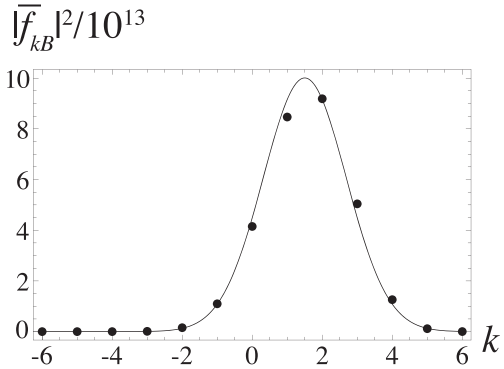

Figure 5. Dots: spectrum of Fourier coefficients (1.9) for superoscillatory function $f_B(x)$ (equation (1.7)), for $a=4$, $N=6$. Full curve: approximation (1.11).

(figure 4). The limiting value exceeds the limiting momentum in the band-limited Fourier expansion (1.1), in which

$$k_{\max}=N,\quad \bar{f}_{kB}=C\sqrt{2\pi}\,\frac{(2N)\,!\,(-1)^{N+k}\left(a^2-1\right)^N}{2^{2N}\,(N+k)\,!\,(N-k)\,!}\left(\frac{a+1}{a-1}\right)^k. \tag{1.9}$$

This is consistent with the behaviour near $x=0$, which elementary expansion of (1.7) shows to be

$$f_B\,(x)\approx C\,\exp\,(\mathrm{i}aNx)\quad (x\approx 0). \tag{1.10}$$

As figure 5 illustrates, the power spectrum in the Fourier range $|k|<1$ is well approximated by

$$\left|\bar{f}_{kB}\right|^2\approx\frac{2Ca^{4N+2}}{k\left(a^2-1\right)}\exp\left\{-\frac{2a^2}{N\left(a^2-1\right)}\left(k-\frac{N}{a}\right)^2\right\}. \tag{1.11}$$

The power spectrum betrays no hint of the superoscillations; these are associated with very small values of $f_B(x)$ arising from destructive interference, which is an effect depending on the phases of the Fourier coefficients, here embodied in the alternating signs in (1.9).

2. Wigner functions

2.1. Wigner formalism

This phase-space representation for functions $f(x)$ is commonly used in quantum or wave physics [14–17]. For non-periodic functions, it can be represented in either of two symmetrically related forms:

$$\begin{aligned} W(x, k) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}y f^*(x+y) f(x-y) \exp(2\mathrm{i}ky) \\ &= \frac{1}{\pi}\int_{-k_{\max}}^{k_{\max}} \mathrm{d}q \bar{f}^*(k+q) \bar{f}(k-q) \exp(-2\mathrm{i}xq) \end{aligned}. \tag{2.1}$$

For 2π-periodic functions, phase space is the cylinder $\{\{-\pi \leqslant x \leqslant \pi\}, \{-\infty < k < \infty\}\}$, and k is discrete and takes integer and half-integer values [18, 19]. The Wigner function can then be written

$$\begin{aligned} W\left(x, k=\frac{1}{2}m\right) &= \frac{1}{2\pi}\sum_{n=-k_{\max}}^{k_{\max}} \bar{f}_n^* \bar{f}_{m-n} \exp(\mathrm{i}x(m-2n)) \\ &= \frac{1}{2\pi}\sum_{l=-k_{\max}}^{k_{\max}} \bar{f}^*_{\frac{1}{2}m-l} \bar{f}_{\frac{1}{2}m+l} \exp(-2\mathrm{i}xl) \quad \left(\frac{1}{2}m + l \text{ integer}\right) \end{aligned}. \tag{2.2}$$

In both cases, W inherits the band-limited property of $f(x)$: $W(x, k) = 0$ if $|k| > k_{\max}$. And elementary arguments, given elsewhere [11], show that the mean momentum k equals the local momentum (1.2):

$$\begin{aligned} k_W(x) &= \frac{\int_{-k_{\max}}^{k_{\max}} \mathrm{d}k k W(x, k)}{\int_{-k_{\max}}^{k_{\max}} \mathrm{d}k W(x, k)} = k_{\mathrm{loc}}(x) \, (\text{non-periodic}) \\ &= \frac{\sum_{m=-2k_{\max}}^{2k_{\max}} \left(\frac{1}{2}m\right) W\left(x, \frac{1}{2}m\right)}{\sum_{m=-2k_{\max}}^{2k_{\max}} W\left(x, \frac{1}{2}m\right)} = k_{\mathrm{loc}}(x) \, (2\pi\text{-periodic}) \end{aligned} \tag{2.3}$$

The Wigner function has been related to superoscillations in an optical context [20].

2.2. Wigner numerics

For the function $f_A(x)$, it is not hard to show that the Wigner function consists of just three terms:

$$\begin{aligned} W_A(x, -1) &= \frac{a^2}{1+a^2-2a\cos 2x}, \\ W_A(x, 0) &= \frac{-2a\cos 2x}{1+a^2-2a\cos 2x}, \\ W_A(x, +1) &= \frac{1}{1+a^2-2a\cos 2x}. \end{aligned} \tag{2.4}$$

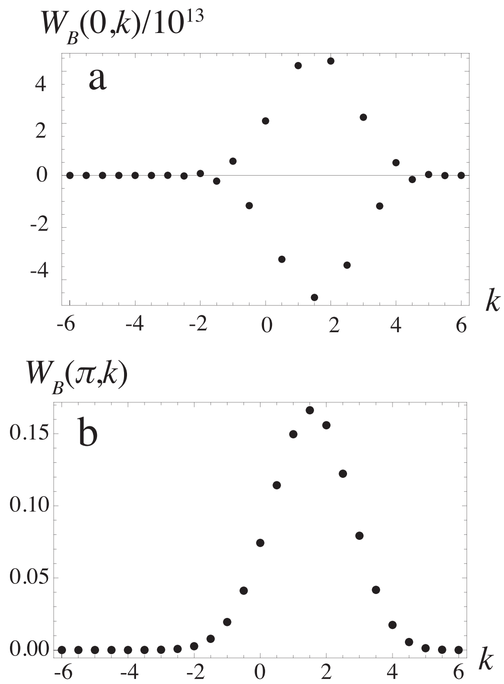

Figure 6. Wigner functions (2.2) as functions of wavenumber k for the function $f_B(x)$ (equation (1.7)) for $a=4$, $N=6$, for (a): $x=0$, (b) $x=\pi$.

(there are no half-integer values in this case), giving the local wavenumber (1.6). The negative values $W_A(x,0)$ for $|x|<\pi/4$ are responsible for the supershift, i.e. $|k_{\mathrm{loc}A}(x)|>1$.

For the function $f_B(x)$, the Wigner function, where k takes integer and half-integer values, for sample values of the parameters a and L, is illustrated in figure 6. Note that W_A takes some negative values where $x=0$ (figure 6(a)), which is the region where f_B superoscillates, but is always positive for $x=\pi$, which is the region where f_B does not superoscillate.

3. Windowed Fourier transforms and Husimi functions

3.1. Husimi formalism

For non-periodic functions $f(y)$, positions near x can be selected by multiplication by a Gaussian window of width L centred on $y=x$, and the Husimi function [21, 22] is the square of the corresponding Fourier transform:

$$H(x,k)=\frac{1}{(2\pi)^{3/2}L}\left|\int_{-\infty}^{\infty}\mathrm{d}y f(y)\exp\left(-\frac{(y-x)^2}{4L^2}\right)\exp(-\mathrm{i}ky)\right|^2. \quad (3.1)$$

This expression reveals a close connection to the weak measurement scheme [8]. k represents the position coordinate of a pointer, x parameterizes the degree of superoscillation, and the Husimi function represents the probability density ($|\text{wavefunction}|^2$) of the pointer, with initial width $1/L$, after a weak measurement (see section 2 of [23], especially equation (2.2)).

An equivalent interpretation of (3.1) [24] is that H is the overlap between f and a coherent state with squeezing represented by L and centred on x, k:

$$H(x,\, k) = \frac{1}{2\pi}\left|\left\langle \psi_{\rm c}(x,\, k)\middle| f\right\rangle\right|^2, \tag{3.2}$$

where the wavefunction of the coherent state is [25]

$$\left\langle y\middle|\psi_{\rm c}(x,\, k)\right\rangle = \frac{1}{\left(2\pi L^2\right)^{1/4}}\exp\left\{ iky - \frac{(x-y)^2}{4L^2}\right\}. \tag{3.3}$$

Yet another interpretation is that H is the Wigner function W, smoothed by a Gaussian in phase space [26]:

$$H(x,\, k) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}{\rm d}y\int_{-k_{\max}}^{k_{\max}}{\rm d}qW(y,\, q)\exp\left\{-\frac{1}{2L^2}(y-x)^2 - 2L^2(q-k)^2\right\}. \tag{3.4}$$

(In physical units, this corresponds to a Gaussian whose phase-space area is Planck's constant h, representing a pure coherent state.)

Obviously, H is positive semi-definite, but it is no longer band-limited. Elementary algebra shows that the corresponding mean momentum is

$$k_H(x) = \frac{\int_{-\infty}^{\infty}{\rm d}kkH(x,\, k)}{\int_{-\infty}^{\infty}{\rm d}kH(x,\, k)} = \frac{\int_{-\infty}^{\infty}{\rm d}y\,|f(y)|^2 k_{\rm loc}(y)\exp\left\{-\frac{(y-x)^2}{2L^2}\right\}}{\int_{-\infty}^{\infty}{\rm d}y\,|f(y)|^2\exp\left\{-\frac{(y-x)^2}{2L^2}\right\}}. \tag{3.5}$$

This is the local wavenumber (1.2), weighted by $|f|^2$ and smoothed by a Gaussian window centred on x. If the window is sufficiently narrow, it is clear that $k_H(x) \approx k_{\rm loc}(x)$. This emergence of $k_{\rm loc}(x)$ for narrow windows is one of our main results. It corresponds, in the alternative explanation described in the Introduction, to the supershift in a weak measurement when the pointer wavefunction (whose width here would be $1/L$) is broad enough [23].

For periodic functions $f(x)$, it is convenient to use a periodic window, namely the sum of translated Gaussians

$$\begin{aligned}\left[\exp\left\{-\tfrac{(x-y)^2}{4L^2}\right\}\right]_{\rm p} &\equiv \sum_{n=-\infty}^{\infty}\exp\left\{-\frac{(x-y-2n\pi)^2}{4L^2}\right\}\\ &= \frac{L}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\exp\left\{-L^2n^2 + {\rm i}n(x-y)\right\}\\ &= \frac{L}{\sqrt{\pi}}\theta_3\left(-\frac{1}{2}(y-x),\, \exp\left(-L^2\right)\right).\end{aligned} \tag{3.6}$$

The second equality is obtained by applying the Poisson summation formula (1.8.14 of [27]), and the third identifies the sum as a Jacobi theta function (20.2.3) of [27]). Then the Husimi function on the phase-space cylinder can be represented in either of the forms

$$\begin{aligned}H(x,\, k) &= \frac{1}{(2\pi)^{3/2}L}\left|\int_{-\pi}^{\pi}{\rm d}y f(y)\exp(-{\rm i}ky)\left[\exp\left(-\tfrac{(y-x)^2}{4L^2}\right)\right]_{\rm p}\right|^2\\ &= \frac{1}{\pi\sqrt{2}}\left|\sum_{n=-k_{\max}}^{k_{\max}}\bar{f}_n\exp\left({\rm i}x(m-k) - L^2(m-k)^2\right)\right|^2.\end{aligned} \tag{3.7}$$

Here k is integer (no half-integer values). And in contrast to the Wigner function W, H does not inherit the band-limited property of $f(x)$: it can (and usually does) take non-zero values for $|k| > k_{\max}$. The local mean momentum is

$$k_H(x) = \frac{\int_{-\infty}^{\infty} \mathrm{d}k k H(x, k)}{\int_{-\infty}^{\infty} \mathrm{d}k H(x, k)} = \frac{\int_{-\pi}^{\pi} \mathrm{d}y \, |f(y)|^2 k_{\mathrm{loc}}(y) \left[\exp\left\{ -\frac{(y-x)^2}{4L^2} \right\} \right]_{\mathrm{p}}^2}{\int_{-\pi}^{\pi} \mathrm{d}y \, |f(y)|^2 \left[\exp\left\{ -\frac{(y-x)^2}{4L^2} \right\} \right]_{\mathrm{p}}^2}, \tag{3.8}$$

differing from (3.5) only by including the periodized window.

3.2. Husimi numerics

For $f_A(x)$, the Husimi function is

$$\begin{aligned} H_A(x, k) = C\Big(& \exp\left(-4L^2(k-1)^2\right) + a^2 \exp\left(-4L^2(k+1)^2\right) \\ & - 2a \exp\left(-4L^2\left(k^2+1\right)\right) \cos(2x) \Big). \end{aligned} \tag{3.9}$$

This is illustrated in figure 7. For $x = 0$ (figures (a), (c), (e)) H_A is centred on the supershifted local wavenumber (1.5), which for this case is $k = 19$, provided L is small enough as in figure (e). This displays the mathematical miracle of the supershift phenomenon in its simplest form: although (3.9) consists of Gaussians at $k = +1$, $k = -1$ and $k = 0$, the sum can be centred on the value $(1 + a)/(1 - a)$ which lies outside this range. (This was the basis of a recent investigation [28]) of the possibility that weak measurment theory might explain a claim (later retracted) that neutrinos had been observed to travel faster than light.) For $x = \pi/2$, where f_A does not superoscillate, there is no supershift.

The Husimi mean momentum can be calculated exactly for this case:

$$\begin{aligned} k_{HA}(x) &= \frac{1-a^2}{1+a^2-2a\exp\left(-4L^2\right)\cos(2x)} \\ &\to \frac{1-a^2}{1+a^2-2a\cos(2x)} \quad (L \to \infty) \\ &\to \frac{1+a}{1-a} \quad (L \to 0). \end{aligned} \tag{3.10}$$

This expression, illustrated in figure 8, shows precisely how the supershift exists for narrow windows, i.e. it emerges for smaller L and is washed away for larger L.

For the function $f_B(x)$, superoscillations and supershifts are much more dramatic. Figure 9 shows the Husimi function as a function of k (taking integer values), for $x = 0$, where f_B superoscillates (figures (a), (c), (e), (g)), and $x = \pi$ (figures (b), (d), (f), (h)), where it does not. In the first case, the supershifts, centred on $k = 24$, emerge clearly as L gets smaller (see especially figure 9(g)), while in the second case the k-distribution stubbornly refuses to supershift, even for very small L (cf figure 9(h)).

The explicit form for the Husimi mean wavenumber for $f_B(x)$ as a function of x is, from (3.8),

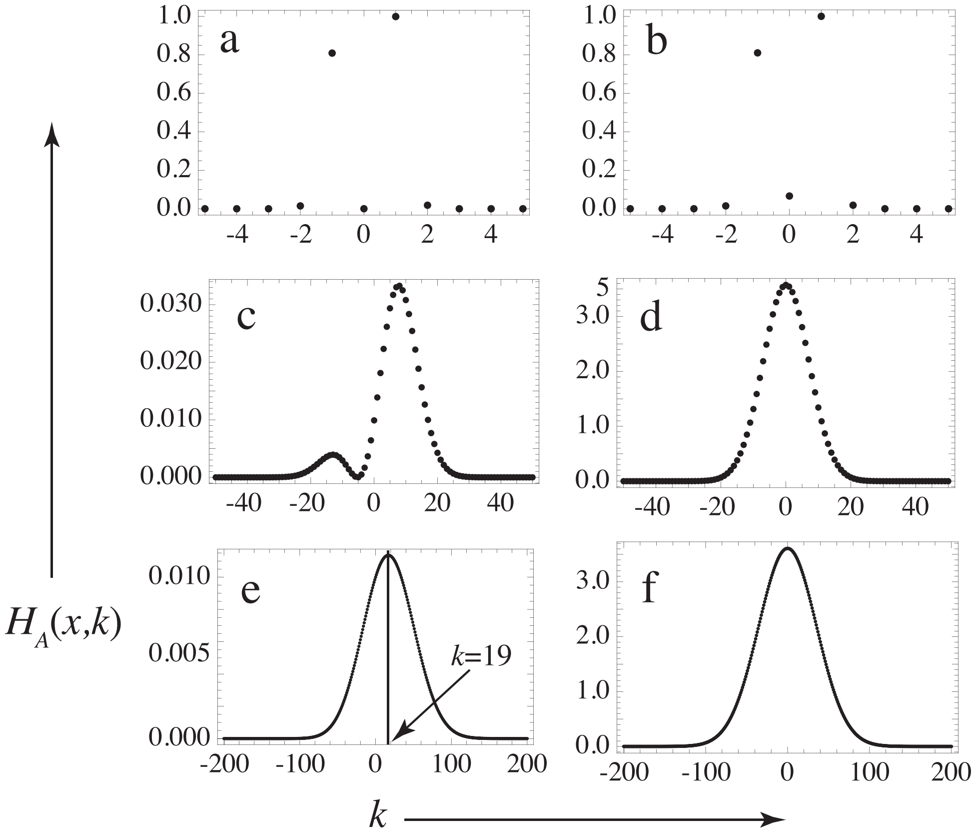

Figure 7. Husimi function (windowed Fourier transform) for $f_A(x)$ (equation (1.4)), for $a = 0.9$ and (a), (c), (e): $x = 0$; (b), (d), (f): $x = \pi/2$; (a), (b): $L = 1$; (c), (d): $L = 0.1$; (e), (f): $L = 0.02$. The supershift appears in (e). (In (e), (f), the individual dots are too close to see.).

$$k_{HB}(x) = aN \frac{\int_{-\pi}^{\pi} \mathrm{d}y \left[\exp\left\{ -\frac{(y-x)^2}{4L^2} \right\} \right]_{\mathrm{p}}^2 \left(\cos\left(\frac{1}{2}x\right)^2 + a^2 \sin\left(\frac{1}{2}x\right)^2 \right)^{2N-1}}{\int_{-\pi}^{\pi} \mathrm{d}y \left[\exp\left\{ -\frac{(y-x)^2}{4L^2} \right\} \right]_{\mathrm{p}}^2 \left(\cos\left(\frac{1}{2}x\right)^2 + a^2 \sin\left(\frac{1}{2}x\right)^2 \right)^{2N}}. \qquad (3.11)$$

This is illustrated in figure 10, and again clearly shows the supershift near $x = 0$ (here corresponding to $k_{\mathrm{loc}} = 24$) emerging as L decreases (figure 10(d)).

4. Concluding remarks

Supershifts associated with superoscillatory functions $f(x)$, described by the local wavenumber $k_{\mathrm{loc}}(x)$, have been well understood physically [8, 10] and also explored mathematically in some detail [4, 5, 29]. The additional insights we have obtained here come from interpreting these phenomena in terms of a phase space, in which x and k are considered simultaneously. An unexpected outcome has been that two ways of representing band-limited finctions $f(x)$ in phase space give two completely different interpretations of the supershift.

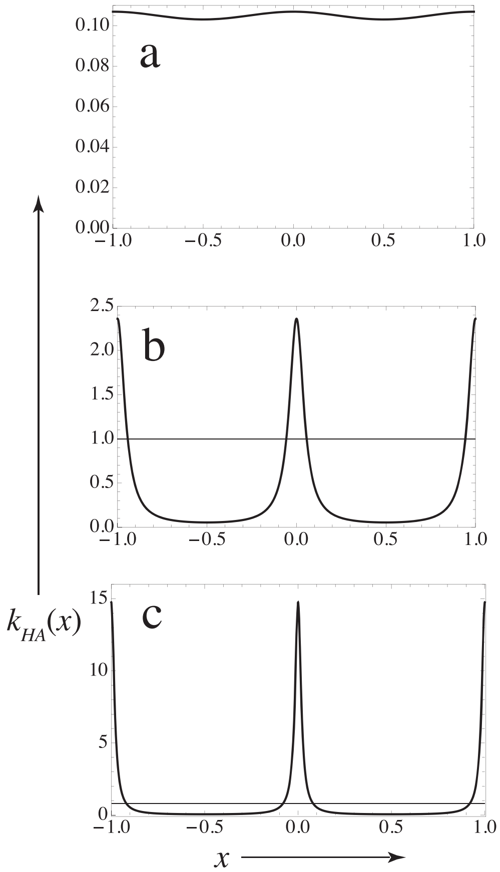

Figure 8. Husimi-calculated local wavenumber (3.10) for $f_A(x)$ for $a = 0.9$ and (a) $L = 1$, (b) $L = 0.1$, (c) $L = 0.02$; the supershift corresponds to $k_{HA} > 1$.

In the Wigner representation, the supershift is possible because $W(x, k)$, although real and band-limited, can be negative for some values of k in regions x where $f(x)$ is superoscillatory. Therefore, as has often been noted [12] W cannot be regarded as a probability distribution in the strict sense. And for a function with negative regions, the mean value can lie outside its domain.

By contrast, the Husimi function $H(x, k)$—which is simply $W(x, k)$ smoothed with a phase-space Gaussian, or, alternatively, the square of the Fourier transform of the Gauss-windowed $f(x)$—is both real and non-negative. But the Gaussian windowing or smoothing means that H is not band-limited even if $f(x)$ is. Then the Husimi-averaged k for fixed x can be large, and if the x window width L is small enough it equals the supershift $k_{\rm loc}(x)$.

These two interpretations are complementary ways of understanding the relation between superoscillations and supershifts, which is itself a kind of Fourier complementarity associated with band-limited functions $f(x)$. It is clear that the quantum weak measurement scheme, involving a set of physical concepts which led to the discovery of these function-theoretic properties, is a continuing source of richness in mathematics as well as physics.

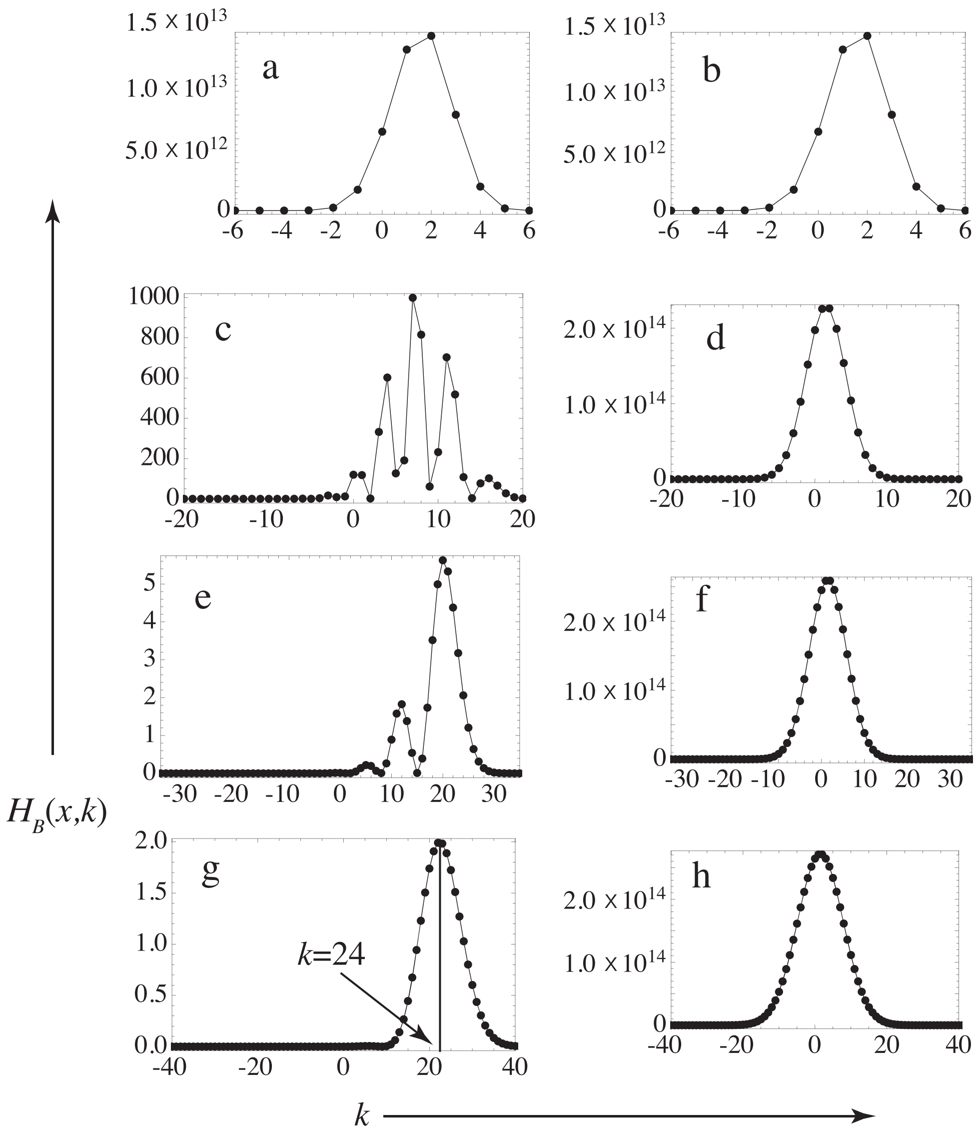

Figure 9. Husimi function (windowed Fourier transform) for $f_B(x)$ (equation (1.7)), for $a=4$, $N=6$ and (a), (c), (e), (g) $x=0$; (b), (d), (f), (h) $x=\pi$; (a), (b), $L=4$; (c), (d), $L=0.2$; (e), (f), $L=0.12$; (g), (h), $L=0.08$.

Acknowlegments

M V Berry thanks the Lewiner Institute, Technion, Israel, for hospitality while this work was done; his research is supported by the Leverhulme Trust. N Moiseyev acknowledges ISF grant 298/11 and Light and Matter I-Core grant for partial support.

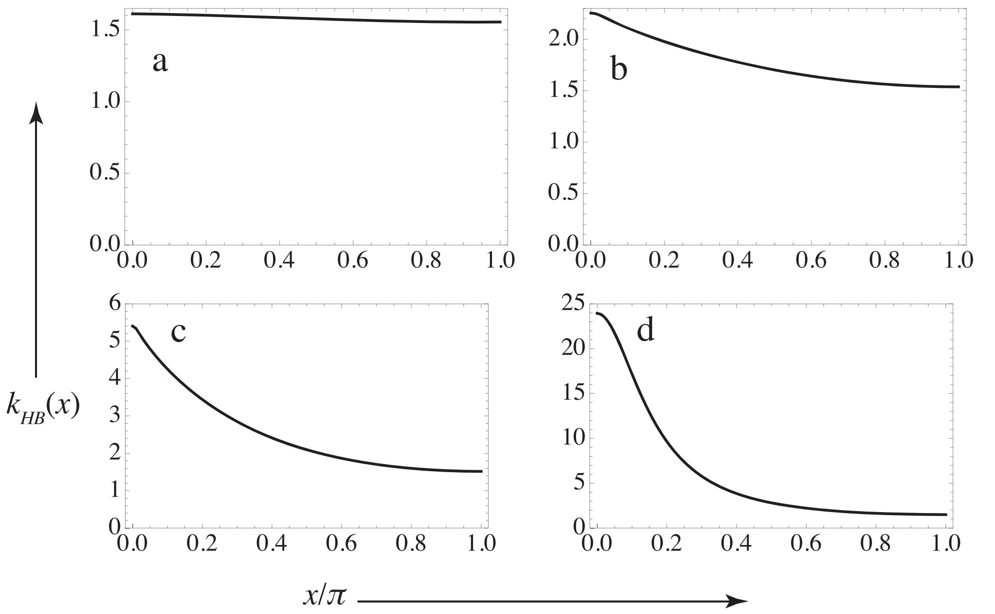

Figure 10. Husimi-calculated local wavenumber (26) for $f_B(x)$ for $a=4$, $N=6$ and (a) $L=1$, (b) $L=0.5$, (c) $L=0.25$, (d) $L=0.025$; supershifts correspond to $k_{HB}>6$.

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