

Anosov property of cyclic $SO_0(2, n)$ -Higgs Bundles (arXiv:2406.08118)

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Anosov Property and Higher (rank) Teichmüller Theory

Classical Teichmüller Theory

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The holonomy representations corresponding to the point in $\mathcal{T}(S)$ are called **Fuchsian** representations. Furthermore, we can view $\mathcal{T}(S)$ as a connected component of the character variety

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Moreover, $\mathcal{T}(S)$ is one of the two connected components which consist entirely of discrete and faithful representations. The other one is $\mathcal{T}(\bar{S})$, where \bar{S} denotes the surface S with the opposite orientation.

Anosov Property of Fuchsian Representation

For a Fuchsian representation $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$, we fix a base point $x_0 = (0, 1) \in \mathbb{H}^2 \cong \tilde{S}$ on the universal cover of S . We have

$$\log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} = d_{\mathbb{H}^2}(x_0, \rho(\gamma)(x_0)),$$

σ_i denotes the i -th singular value.

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and so on.

In general, the set of discrete and faithful representations is only closed and NOT open.

Higher Teichmüller Spaces

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- (Hitchin, 1992; Labourie, 2006; Fock–Goncharov, 2006) Hitchin components $\mathcal{T}_{Hit}(S, G)$ for **real split** G , such as $\text{SL}(n, \mathbb{R})$, $\text{SO}_0(n, n + 1)$, $\text{Sp}(2n, \mathbb{R})$ and so on;

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When $G = \text{SL}(2, \mathbb{R})$, the above components coincide with the classical Teichmüller space.

Anosov Property

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The original definition is highly dynamical and we will use an interpreted definition (which is proven equivalent to the original one by Kapovich–Leeb–Porti and Bochi–Potrie–Sambarino).

Also, we will assume G is a semi-simple Lie subgroup of $SL(n, \mathbb{C})$ here to avoid involving a Lie-theoretic description.

Definition

$\rho: \pi_1(S) \rightarrow G$ is P_k -**Anosov** if there exist constants $D, L > 0$ such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geq D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(S),$$

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For $g \in \mathrm{SO}_0(2, n) (n > 2)$, it is remarkable that $\sigma_i(g) = (\sigma_{n+3-i}(g))^{-1}$ and

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Anosov \implies discrete + faithful.

Anosov property is open but not closed. However, all representations in the known higher Teichmüller spaces are Anosov.

Non-Abelian Hodge Correspondence

Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure on S such that it becomes a Riemann surface X . Let \mathcal{K}_X be its canonical line bundle.

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Definition

A $(GL(n, \mathbb{C})\text{-})$ **Higgs bundle** over X is a pair (\mathcal{E}, Φ) consisting of the following data:

- a holomorphic vector bundle \mathcal{E} over X with $\text{rank}(\mathcal{E}) = n$;
- a holomorphic section $\Phi \in H^0(X, \text{End}(\mathcal{E}) \otimes \mathcal{K}_X)$ called **Higgs field**.

The non-Abelian Hodge correspondence exhibit a homeomorphism between the following moduli spaces:

$$\begin{array}{ccc}
 \{\text{reductive representation } \rho: \pi_1(S) \rightarrow \mathrm{GL}(n, \mathbb{C})\} / \mathrm{GL}(n, \mathbb{C}) & & \\
 \text{associated bundle} \downarrow & \uparrow & \text{holonomy} \\
 \{\text{reductive flat vector bundle } (E \rightarrow S, \nabla) \text{ with } \mathrm{rank}(E) = n\} / \mathcal{G} & & \\
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 \end{array}$$

If we equip additional structure on these objects, we can get the non-Abelian Hodge correspondence for general reductive Lie group G .

Kobayashi–Hitchin Correspondence (from Higgs to flat)

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Theorem (Hitchin–Simpson)

If (\mathcal{E}, Φ) is a polystable Higgs bundle with $\deg(\mathcal{E}) = 0$, then there exists an Hermitian metric h on \mathcal{E} such that

$$F(\nabla^h) + [\Phi, \Phi^{*h}] = 0, \quad (1)$$

*where ∇^h is the Chern connection of the metric h , $F(\nabla^h)$ denotes its curvature form and Φ^{*h} is the adjoint of Φ with respect to h . Moreover, if (\mathcal{E}, Φ) is stable, then such h is unique up to a constant scalar.*

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If h solves (1), then

$$\nabla^h + \Phi + \Phi^{*h}$$

gives a flat connection.

From Higgs Bundle to Anosov Representation

Higgs bundle $\xrightarrow{\text{Hitchin's self-dual equation}}$ Anosov property?

Example: Hitchin Component in Higgs Bundle Viewpoint

Let us fix a square root $\mathcal{K}_X^{1/2}$ of \mathcal{K}_X , then the Hitchin component for $\mathrm{SL}(n, \mathbb{R})$ consisting of entirely the Higgs bundles of the following form:

$$\mathcal{E} = \mathcal{K}_X^{(n-1)/2} \oplus \mathcal{K}_X^{(n-3)/2} \oplus \cdots \oplus \mathcal{K}_X^{(1-n)/2},$$

$$\Phi = \begin{pmatrix} 0 & q_2 & q_3 & q_4 & \cdots & q_n \\ 1 & 0 & q_2 & q_3 & \ddots & q_{n-1} \\ & 1 & 0 & q_2 & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & 0 & q_2 \\ & & & & 1 & 0 \end{pmatrix},$$

where $1: \mathcal{K}_X^{(n-1)/2-i} \rightarrow \mathcal{K}_X^{(n-1)/2-(i+1)} \otimes \mathcal{K}_X$ is the natural isomorphism and $q_i \in H^0(X, \mathcal{K}_X^i)$.

It corresponds to the component containing the embedding of Fuchsian representations through the unique irreducible $\mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{SL}(n, \mathbb{R})$.

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- (Z. 2024) α_1 -cyclic $SO_0(2, n)$ -Higgs bundle;
- (Bronstein–Davalò, 2025) a slice of deformations of Barbot representations.

Special $\mathrm{SO}_0(2, 3)$ -Higgs Bundles

Below we consider the Higgs bundle whose underlying bundle is

$$\mathcal{E} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2,$$

where \mathcal{L}_i are line bundles with $\mathcal{L}_i \cong \mathcal{L}_{-i}^\vee$ and $\mathcal{L}_0 \cong \mathcal{O}$. Note that there is a natural pairing on \mathcal{E} defined by

$$Q = \begin{pmatrix} & & & & -1 \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix}$$

Suppose that the Higgs field Φ projects to 0 in $H^0(X, \mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j) \otimes \mathcal{K}_X)$ for any i, j have the same parity and Φ is compatible with Q , then polystable (\mathcal{E}, Φ) gives an $\mathrm{SO}_0(2, 3)$ -representation.

In addition, if (\mathcal{E}, Φ) comes from a variation of Hodge structure, then we have

$$(\mathcal{E}, \Phi) = \mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{L}_2 .$$

Such Higgs bundle is maximal if and only if β is an isomorphism, and maximal representations are known to be Anosov. From a different starting point, S. Filip considered the Higgs bundle of the same form, but instead α is an isomorphism.

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Theorem (Filip, 2021)

A stable $SO_0(2, 3)$ -Higgs bundle of the form

$$\mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{L}_2$$

with α is an isomorphism gives a P_2 -Anosov representation.

Filip proved this theorem by an **analytic method**. Inspired by his method and with some simplification, we extend his results and discover the Anosov property of a general family of $SO_0(2, 3)$ -Higgs bundles.

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Remark

The trivial line bundle \mathcal{L}_0 above can be replaced by an orthogonal vector bundle of rank n to get an $SO(2, n + 2)$ -Higgs bundle. With the assumption of stability, the Anosov property still holds.

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It suffices to show that there exist constants $C_1, C_2, \varepsilon > 0$ such that

$$\|v(x)\|^2 \geq C_1 \cdot \exp(\varepsilon \cdot d_{\mathbb{H}^2}(x, x_0)) - C_2,$$

then the corresponding ρ is P_2 -Anosov.

Filip's estimate (simplified)

Lemma (Filip, 2021)

Given a complete Riemannian manifold (M, g) with the distance function $d: M \times M \rightarrow \mathbb{R}$ and a smooth function $f \in C^\infty(M; \mathbb{R})$. Suppose that f satisfies

(S1) f is a non-negative function with all zeroes isolated;

(S2) $\|df\|_g \gtrsim f$;

(S3) $\|df\|_g \gtrsim f^{1/2}$.

Then f achieves its unique minimum at x_{\min} and there exist constants $C_1, C_2, \varepsilon > 0$, such that

$$f(x) \geq C_1 \cdot \exp(\varepsilon \cdot d(x_{\min}, x)) - C_2, \forall x \in M.$$

A global flat section v decomposes as $\sum_{i=-2}^2 v_i$ with $v_i: \mathbb{H}^2 \rightarrow \mathcal{L}_i$. Choose a real positive initial vector $v \in \mathcal{E}_{x_0}$, i.e. $2\|v_1\|^2 - (\|v_0\|^2 + 2\|v_2\|^2) > 0$ and we take $f_v := \|v_2\|^2$ for the lemma above.

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For Filip's case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\| \gtrsim \max\{\|v_2\|, 1\}$.

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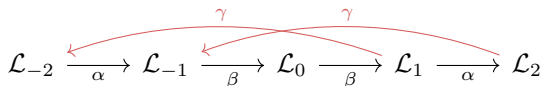
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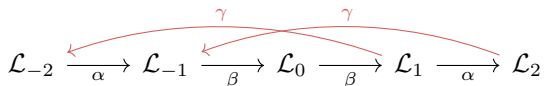
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Non-compact case

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A stable α_1 -cyclic parabolic $SO_0(2, 3)$ -Higgs bundle gives a P_2 -almost-dominated representation through the non-Abelian Hodge correspondence.

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Key point: Set suitable weight such that

$$\|\alpha\| - \|\gamma\| > C.$$

Thank you!