Anosov property of cyclic $\mathrm{SO}_0(2,n)$-Higgs Bundles

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Classical Teichmüller Theory

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$$\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\!\!/\mathrm{PSL}(2,\mathbb{R}).$$

Moreover, $\mathcal{T}(S)$ is one of the two connected components which consist entirely of discrete and faithful representations. The other one is $\mathcal{T}(\overline{S})$, where $\overline{S}$ denotes the surface S with the opposite orientation.

Anosov Property of Fuchsian Representation

For a Fuchsian representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, we fix a base point $x_0 = (0,1) \in \mathbb{H}^2 \cong \widetilde{S}$ on the universal cover of S. We have

$$\log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} = d_{\mathbb{H}^2}(x_0, \rho(\gamma)(x_0)),$$

σ_i denotes the i-th singular value.

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and so on.

In general, the set of discrete and faithful representations is only closed and NOT open.

Higher Teichmüller Spaces

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- (Hitchin, 1992; Labourie, 2006; Fock–Goncharov,2006) Hitchin components $\mathcal{T}_{Hit}(S, G)$ for **real split** G, such as $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SO}_0(n, n+1)$, $\mathrm{Sp}(2n, \mathbb{R})$ and so on;

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- (Burger–Iozzi–Wienhard, 2003) Maximal components $\mathcal{T}_{max}(S, G)$ for **Hermitian** G, such as $\mathrm{SU}(p, q)$, $\mathrm{SO}_0(2, n)$ and so on.

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When $G = \mathrm{SL}(2, \mathbb{R})$, the above components coincide with the classical Teichmüller space.

Anosov Property

Anosov property plays an important role in higher Teichmüller theory. Roughly speaking, the Anosov property introduced by F. Labourie means that the representation $\rho\colon \pi_1(S) \to G$ satisfies some special dynamic properties.

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The original definition is highly dynamical and we will use an interpreted definition (which is proven equivalent to the original one by Kapovich–Leeb–Porti and Bochi–Potrie–Sambarino).

Also, we will assume G is a semi-simple Lie subgroup of $\mathrm{SL}(n, \mathbb{C})$ here to avoid involving a Lie-theoretic description.

Definition

$\rho\colon \pi_1(S) \to G$ is P_k-**Anosov** if there exist constants $D, L > 0$ such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(S),$$

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For $g \in \mathrm{SO}_0(2,n)(n > 2)$, it is remarkable that $\sigma_i(g) = (\sigma_{n+3-i}(g))^{-1}$ and

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Anosov $\Longrightarrow$ discrete + faithful.

Anosov property is open but not closed. However, all representations in the known higher Teichmüller spaces are Anosov.

Non-Abelian Hodge Correspondence

Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure on S such that it becomes a Riemann surface X. Let $\mathcal{K}_X$ be its canonical line bundle.

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Definition

A **($\mathrm{GL}(n,\mathbb{C})$-)Higgs bundle** over X is a pair $(\mathcal{E},\Phi)$ consisting of the following data:

- a holomorphic vector bundle $\mathcal{E}$ over X with $\mathrm{rank}(\mathcal{E})=n$;
- a holomorphic section $\Phi\in\mathrm{H}^0(X,\mathrm{End}(\mathcal{E})\otimes\mathcal{K}_X)$ called **Higgs field**.

The non-Abelian Hodge correspondence exhibit a homeomorphism between the following moduli spaces:

$$\begin{tikzcd}
\{\textbf{reductive}\text{ representation } \rho\colon \pi_1(S) \to \mathrm{GL}(n,\mathbb{C})\}/\mathrm{GL}(n,\mathbb{C}) \arrow[d, shift right=2, "\text{associated bundle}"'] \\
\{\textbf{reductive}\text{ flat vector bundle } (E \to S, \nabla) \text{ with } \operatorname{rank}(E) = n\}/\mathcal{G} \arrow[u, shift right=2, "\text{holonomy}"'] \\
\textbf{harmonic metric} \arrow[u] \arrow[d] \\
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If we equip additional structure on these objects, we can get the non-Abelian Hodge correspondence for general reductive Lie group G.

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Theorem (Hitchin–Simpson)

If $(\mathcal{E}, \Phi)$ *is a polystable Higgs bundle with* $\deg(\mathcal{E}) = 0$, *then there exists an Hermitian metric* h *on* $\mathcal{E}$ *such that*

$$F(\nabla^h) + [\Phi, \Phi^{*_h}] = 0, \qquad (1)$$

where ∇^h *is the Chern connection of the metric* h, $F(\nabla^h)$ *denotes its curvature form and* Φ^{*_h} *is the adjoint of* Φ *with respect to* h. *Moreover, if* $(\mathcal{E}, \Phi)$ *is stable, then such* h *is unique up to a constant scalar.*

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If h solves (1), then

$$\nabla^h + \Phi + \Phi^{*_h}$$

gives a flat connection.

From Higgs Bundle to Anosov Representation

$$\text{Higgs bundle} \xrightarrow{\textbf{Hitchin's self-dual equation}} \text{Anosov property?}$$

Example: Hitchin Component in Higgs Bundle Viewpoint

Let us fix a square root $\mathcal{K}_X^{1/2}$ of $\mathcal{K}_X$, then the Hitchin component for $\mathrm{SL}(n,\mathbb{R})$ consisting of entirely the Higgs bundles of the following form:

$$\mathcal{E} = \mathcal{K}_X^{(n-1)/2} \oplus \mathcal{K}_X^{(n-3)/2} \oplus \cdots \oplus \mathcal{K}_X^{(1-n)/2},$$

$$\Phi = \begin{pmatrix} 0 & q_2 & q_3 & q_4 & \cdots & q_n \\ 1 & 0 & q_2 & q_3 & \ddots & q_{n-1} \\ & 1 & 0 & q_2 & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & 0 & q_2 \\ & & & & 1 & 0 \end{pmatrix},$$

where $1\colon \mathcal{K}_X^{(n-1)/2-i} \to \mathcal{K}_X^{(n-1)/2-(i+1)} \otimes \mathcal{K}_X$ is the natural isomorphism and $q_i \in \mathrm{H}^0(X, \mathcal{K}_X^i)$.

It corresponds to the component containing the embedding of Fuchsian representations through the unique irreducible $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$.

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- (Bronstein–Davalo, 2025) a slice of deformations of Barbot representations.

Special $\mathrm{SO}_0(2,3)$-Higgs Bundles

Below we consider the Higgs bundle whose underlying bundle is

$$\mathcal{E} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2,$$

where $\mathcal{L}_i$ are line bundles with $\mathcal{L}_i \cong \mathcal{L}_{-i}^\vee$ and $\mathcal{L}_0 \cong \mathcal{O}$. Note that there is a natural pairing on $\mathcal{E}$ defined by

$$Q = \begin{pmatrix} & & & & -1 \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix}$$

Suppose that the Higgs field Φ projects to 0 in $\mathrm{H}^0(X, \mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j) \otimes \mathcal{K}_X)$ for any i, j have the same parity and Φ is compatible with Q, then polystable $(\mathcal{E}, \Phi)$ gives an $\mathrm{SO}_0(2,3)$-representation.

In addition, if $(\mathcal{E}, \Phi)$ comes from a variation of Hodge structure, then we have

$$(\mathcal{E}, \Phi) = \mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{L}_2 \ .$$

Such Higgs bundle is maximal if and only if β is an isomorphism, and maximal representations are known to be Anosov. From a different starting point, S. Filip considered the Higgs bundle of the same form, but instead α is an isomorphism.

In addition, if $(\mathcal{E}, \Phi)$ comes from a variation of Hodge structure, then we have

$$(\mathcal{E}, \Phi) = \begin{tikzcd} \mathcal{L}_{-2} \arrow[r, "\alpha"] & \mathcal{L}_{-1} \arrow[r, "\beta"] & \mathcal{L}_0 \arrow[r, "\beta"] & \mathcal{L}_1 \arrow[r, "\alpha"] & \mathcal{L}_2 \end{tikzcd} .$$

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Theorem (Filip, 2021)

A stable $\mathrm{SO}_0(2,3)$*-Higgs bundle of the form*

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with α *is an isomorphism gives a* P_2*-Anosov representation.*

Filip proved this theorem by an **analytic method**. Inspired by his method and with some simplification, we extend his results and discover the Anosov property of a general family of $\mathrm{SO}_0(2,3)$-Higgs bundles.

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Remark

The trivial line bundle $\mathcal{L}_0$ *above can be replaced by an orthogonal vector bundle of rank* n *to get an* $\mathrm{SO}(2,n+2)$*-Higgs bundle. With the assumption of stability, the Anosov property still holds.*

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It suffices to show that there exist constants $C_1, C_2, \varepsilon > 0$ such that

$$\|v(x)\|^2 \geqslant C_1 \cdot \exp(\varepsilon \cdot d_{\mathbb{H}^2}(x, x_0)) - C_2,$$

then the corresponding ρ is P_2-Anosov.

Filip's estimate (simplified)

Lemma (Filip, 2021)

Given a complete Riemannian manifold (M, g) *with the distance function* $d\colon M \times M \to \mathbb{R}$ *and a smooth function* $f \in C^\infty(M;\mathbb{R})$. *Suppose that* f *satisfies*

(S1) f *is a non-negative function with all zeroes isolated;*

(S2) $\|\mathrm{d}f\|_g \gtrsim f$*;*

(S3) $\|\mathrm{d}f\|_g \gtrsim f^{1/2}$*.*

Then f *achieves its unique minimum at* $x_{\min}$ *and there exist constants* $C_1, C_2, \varepsilon > 0$*, such that*

$$f(x) \geqslant C_1 \cdot \exp(\varepsilon \cdot d(x_{\min}, x)) - C_2, \forall x \in M.$$

A global flat section v decomposes as $\sum_{i=-2}^{2} v_i$ with $v_i \colon \mathbb{H}^2 \to \mathcal{L}_i$. Choose a real positive initial vector $v \in \mathcal{E}_{x_0}$, i.e. $2\|v_1\|^2 - \left(\|v_0\|^2 + 2\|v_2\|^2\right) > 0$ and we take $f_v := \|v_2\|^2$ for the lemma above.

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For Filip's case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\| \gtrsim \max\{\|v_2\|, 1\}$.

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For our case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\| \gtrsim \max\{\|v_2\|, 1\}$.

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Non-compact case

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A stable α_1-cyclic parabolic $\mathrm{SO}_0(2,3)$-Higgs bundle gives a P_2-almost-dominated representation through the non-Abelian Hodge correspondence.

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Key point: Set suitable weight such that

$$||\alpha|| - ||\gamma|| > C.$$

Thank you!