Anosov property of cyclic $\mathrm{SO}_0(2,3)$-Higgs Bundles
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Anosov Property and Higher (rank) Teichmüller Theory

Classical Teichmüller Theory

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$$\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R})) /\!\!/ \mathrm{PSL}(2, \mathbb{R}).$$

Moreover, $\mathcal{T}(S)$ is one of the two connected components which consist entirely of discrete and faithful representations. The other one is $\mathcal{T}(\overline{S})$, where $\overline{S}$ denotes the surface S with the opposite orientation.

Anosov Property of Fuchsian Representation

For a Fuchsian representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, we fix a base point $x_0 = (0,1) \in \mathbb{H}^2 \cong \widetilde{S}$ on the universal cover of S. We have

$$\log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} = d_{\mathbb{H}^2}(x_0, \rho(\gamma)(x_0)),$$

σ_i denotes the i-th singular value.

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and so on.

In general, the set of discrete and faithful representations is only closed and NOT open.

Higher Teichmüller Spaces

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- (Hitchin, 1992; Labourie, 2006; Fock–Goncharov,2006) Hitchin components $\mathcal{T}_{Hit}(S, G)$ for **real split** G, such as $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SO}_0(n, n+1)$, $\mathrm{Sp}(2n, \mathbb{R})$ and so on;

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- (Burger–Iozzi–Wienhard, 2003) Maximal components $\mathcal{T}_{max}(S, G)$ for **Hermitian** G, such as $\mathrm{SU}(p, q)$, $\mathrm{SO}_0(2, n)$ and so on.

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When $G = \mathrm{SL}(2, \mathbb{R})$, the above components coincide with the classical Teichmüller space.

Anosov Property

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The original definition is highly dynamical and we will use an interpreted definition (which is proven equivalent to the original one by Kapovich–Leeb–Porti and Bochi–Potrie–Sambarino).

Also, we will assume G is a semi-simple Lie subgroup of $\mathrm{SL}(n, \mathbb{C})$ here to avoid involving a Lie-theoretic description.

Definition

$\rho\colon \pi_1(S) \to G$ is P_k-**Anosov** if there exist constants $D, L > 0$ such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(S),$$

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For $g \in \mathrm{SO}_0(2, n)(n > 2)$, it is remarkable that $\sigma_i(g) = (\sigma_{n+3-i}(g))^{-1}$ and $\sigma_3(g) = \cdots = \sigma_n(g) = 1$.

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Anosov property is open but not closed. However, all representations in the known higher Teichmüller spaces are Anosov.

Non-Abelian Hodge Correspondence

Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure on S such that it becomes a Riemann surface X. Let $\mathcal{K}_X$ be its canonical line bundle.

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Definition

A **($\mathrm{GL}(n,\mathbb{C})$-)Higgs bundle** over X is a pair $(\mathcal{E}, \Phi)$ consisting of the following data:

- a holomorphic vector bundle $\mathcal{E}$ over X with $\operatorname{rank}(\mathcal{E}) = n$;
- a holomorphic section $\Phi \in \mathrm{H}^0(X, \operatorname{End}(\mathcal{E}) \otimes \mathcal{K}_X)$ called **Higgs field**.

The non-Abelian Hodge correspondence exhibit a homeomorphism between the following moduli spaces:

$$\{\textbf{reductive representation } \rho\colon \pi_1(S) \to \mathrm{GL}(n,\mathbb{C})\}/\mathrm{GL}(n,\mathbb{C})$$

$$\text{associated bundle} \big\downarrow \quad \big\uparrow \text{holonomy}$$

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$$

If we equip additional structure on these objects, we can get the non-Abelian Hodge correspondence for general reductive Lie group G.

Kobayashi–Hitchin Correspondence (from Higgs to flat)

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Theorem (Hitchin–Simpson)

If $(\mathcal{E}, \Phi)$ is a polystable Higgs bundle with $\deg(\mathcal{E}) = 0$, then there exists an Hermitian metric h on $\mathcal{E}$ such that

$$F(\nabla^h) + [\Phi, \Phi^{*_h}] = 0, \tag{1}$$

where ∇^h is the Chern connection of the metric h, $F(\nabla^h)$ denotes its curvature form and Φ^{*_h} is the adjoint of Φ with respect to h. Moreover, if $(\mathcal{E}, \Phi)$ is stable, then such h is unique up to a constant scalar.

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If h solves (1), then

$$\nabla^h + \Phi + \Phi^{*_h}$$

gives a flat connection.

From Higgs Bundle to Anosov Representation

$$\text{Higgs bundle} \xrightarrow{\textbf{Hitchin's self-dual equation}} \text{Anosov property?}$$

Example: Hitchin Component in Higgs Bundle Viewpoint

Let us fix a square root $\mathcal{K}_X^{1/2}$ of $\mathcal{K}_X$, then the Hitchin component for $\mathrm{SL}(n,\mathbb{R})$ consisting of entirely the Higgs bundles of the following form:

$$\mathcal{E} = \mathcal{K}_X^{(n-1)/2} \oplus \mathcal{K}_X^{(n-3)/2} \oplus \cdots \oplus \mathcal{K}_X^{(1-n)/2},$$

$$\Phi = \begin{pmatrix} 0 & q_2 & q_3 & q_4 & \cdots & q_n \\ 1 & 0 & q_2 & q_3 & \ddots & q_{n-1} \\ & 1 & 0 & q_2 & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & 0 & q_2 \\ & & & & 1 & 0 \end{pmatrix},$$

where $1 \colon \mathcal{K}_X^{(n-1)/2-i} \to \mathcal{K}_X^{(n-1)/2-(i+1)} \otimes \mathcal{K}_X$ is the natural isomorphism and $q_i \in \mathrm{H}^0(X, \mathcal{K}_X^i)$.
It corresponds to the component containing the embedding of Fuchsian representations through the unique irreducible $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$.

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- (Z. 2024) α_1-cyclic $\mathrm{SO}_0(2, 3)$-Higgs bundle;
- (Bronstein–Davalo, 2025) a slice of deformations of Barbot representations.

Special $\mathrm{SO}_0(2,3)$-Higgs Bundles

Below we consider the Higgs bundle whose underlying bundle is

$$\mathcal{E} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2,$$

where $\mathcal{L}_i$ are line bundles with $\mathcal{L}_i \cong \mathcal{L}_{-i}^\vee$ and $\mathcal{L}_0 \cong \mathcal{O}$. Note that there is a natural pairing on $\mathcal{E}$ defined by

$$Q = \begin{pmatrix} & & & & -1 \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix}$$

Suppose that the Higgs field Φ projects to 0 in $\mathrm{H}^0(X, \mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j) \otimes \mathcal{K}_X)$ for any i, j have the same parity and Φ is compatible with Q, then polystable $(\mathcal{E}, \Phi)$ gives an $\mathrm{SO}_0(2,3)$-representation.

In addition, if $(\mathcal{E}, \Phi)$ comes from a variation of Hodge structure, then we have

$$(\mathcal{E}, \Phi) = \mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{L}_2 .$$

Such Higgs bundle is maximal if and only if β is an isomorphism, and maximal representations are known to be Anosov. From a different starting point, S. Filip considered the Higgs bundle of the same form, but instead α is an isomorphism.

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Theorem (Filip, 2021)

A stable $\mathrm{SO}_0(2,3)$-Higgs bundle of the form

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with α is an isomorphism gives a P_2-Anosov representation.

Filip proved this theorem by an **analytic method**. Inspired by his method and with some simplification, we extend his results and discover the Anosov property of a general family of $\mathrm{SO}_0(2,3)$-Higgs bundles.

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Remark

The trivial line bundle $\mathcal{L}_0$ above can be replaced by an orthogonal vector bundle of rank n to get an $\mathrm{SO}(2,n+2)$-Higgs bundle. With the assumption of stability, the Anosov property still holds.

Sketch of Proof

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It suffices to show that there exist constants $C_1, C_2, \varepsilon > 0$ such that

$$\|v(x)\|^2 \geqslant C_1 \cdot \exp(\varepsilon \cdot d_{\mathbb{H}^2}(x, x_0)) - C_2,$$

then the corresponding ρ is P_2-Anosov.

Filip's estimate (simplified)

Lemma (Filip, 2021)

Given a complete Riemannian manifold (M, g) *with the distance function* $d\colon M \times M \to \mathbb{R}$ *and a smooth function* $f \in C^\infty(M; \mathbb{R})$*. Suppose that* f *satisfies*

(S1) f *is a non-negative function with all zeroes isolated;*

(S2) $\|\mathrm{d}f\|_g \gtrsim f$*;*

(S3) $\|\mathrm{d}f\|_g \gtrsim f^{1/2}$*.*

Then f *achieves its unique minimum at* $x_{\min}$ *and there exist constants* $C_1, C_2, \varepsilon > 0$*, such that*

$$f(x) \geqslant C_1 \cdot \exp(\varepsilon \cdot d(x_{\min}, x)) - C_2, \forall x \in M.$$

A global flat section v decomposes as $\sum_{i=-2}^{2} v_i$ with $v_i\colon \mathbb{H}^2 \to \mathcal{L}_i$. Choose a real positive initial vector $v \in \mathcal{E}_{x_0}$, i.e. $2\|v_1\|^2 - \left(\|v_0\|^2 + 2\|v_2\|^2\right) > 0$ and we take $f_v := \|v_2\|^2$ for the lemma above.

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For Filip's case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\| \gtrsim \max\{\|v_2\|, 1\}$.

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- (S1) $\cancel{\Longleftarrow}$ v_2 is a ~~holomorphic~~ section.

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For our case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\| \gtrsim \max\{\|v_2\|, 1\}$.

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Non-compact case

Lots of results in this section can be generalized when the closed hyperbolic Riemann surface X is replaced by a hyperbolic Riemann surface $X := \overline{X} \setminus D$ of finite type, i.e. both $g(\overline{X})$ and $\#D$ are finite.

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A stable α_1-cyclic parabolic $\mathrm{SO}_0(2,3)$-Higgs bundle gives a P_2-almost-dominated representation through the non-Abelian Hodge correspondence.

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Key point: Set suitable weight such that

$$||\alpha|| - ||\gamma|| > C.$$

Thank you!