Color Refinement for Relational Structures

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What the next 20 minutes will be about

Color Refinement

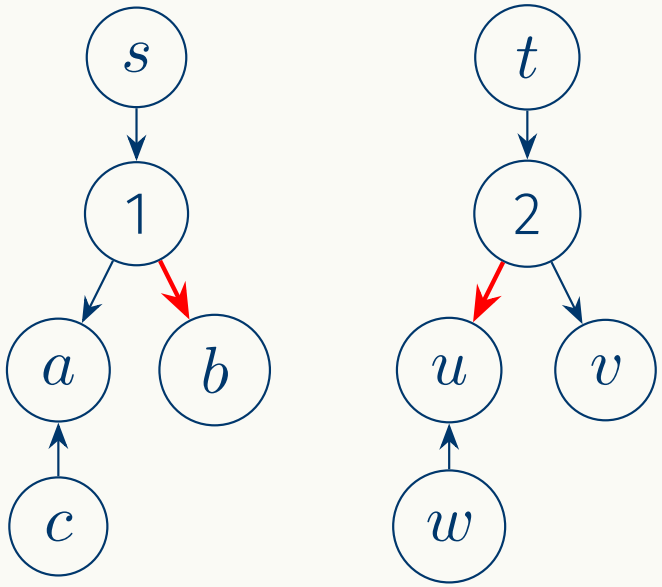
How to lift CR to Relational Structures

Doing it Efficiently

Simpler Approaches?

Outlook

Color Refinement: Iteratively compute stable coloring γ_∞



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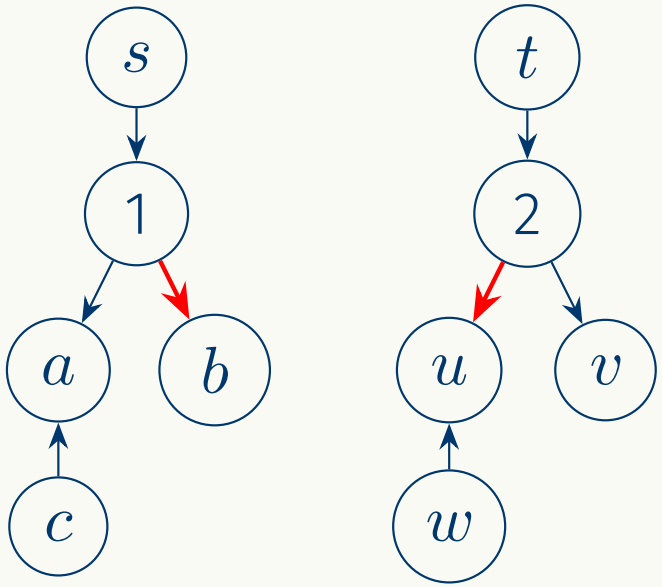

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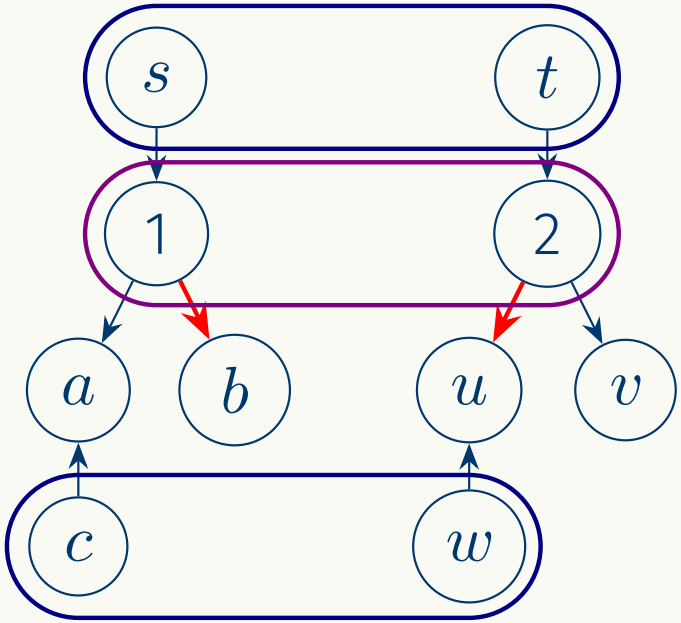

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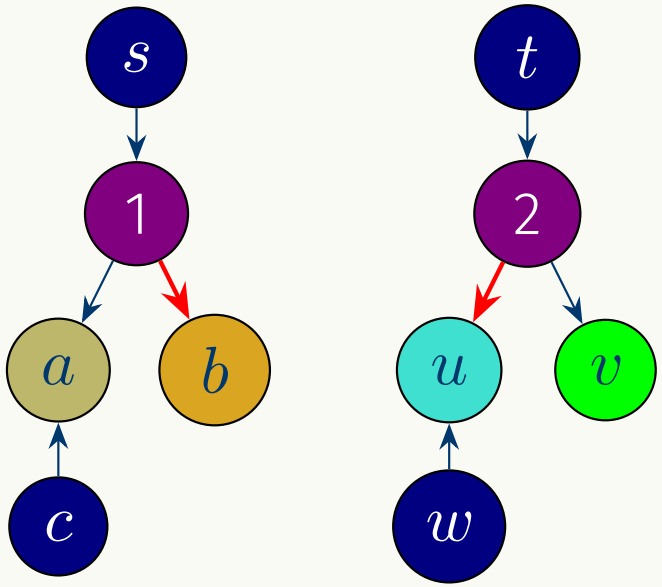

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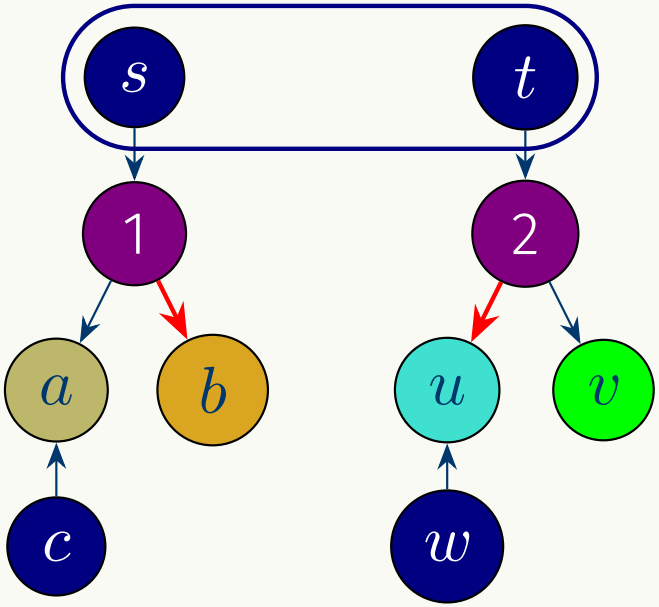

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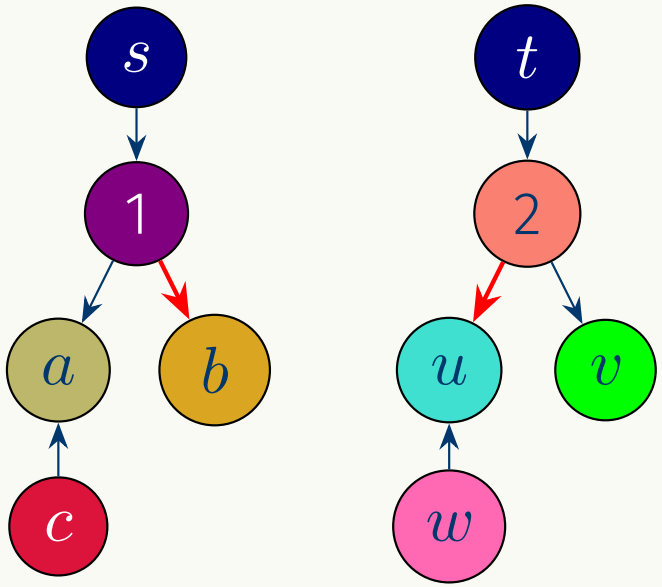

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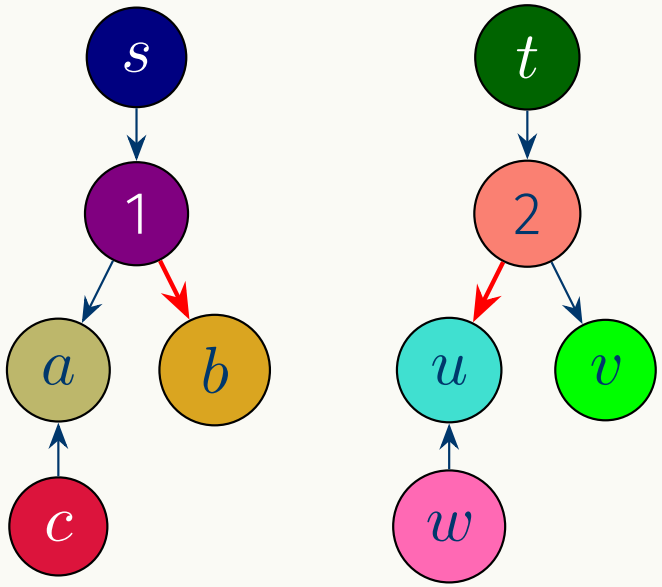

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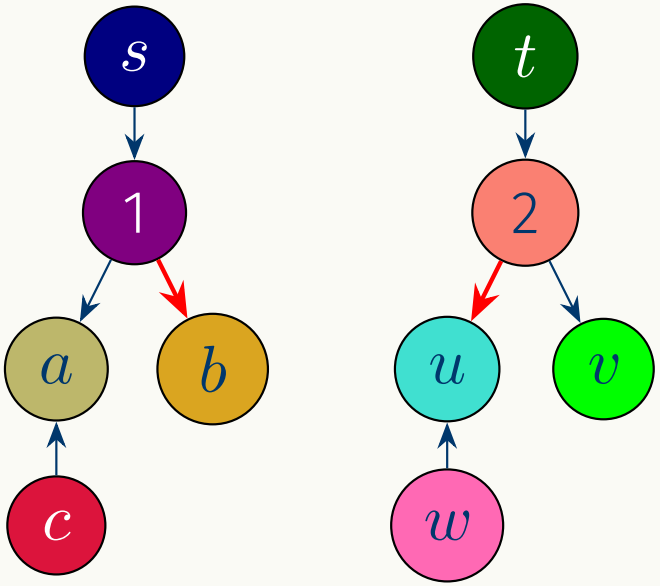

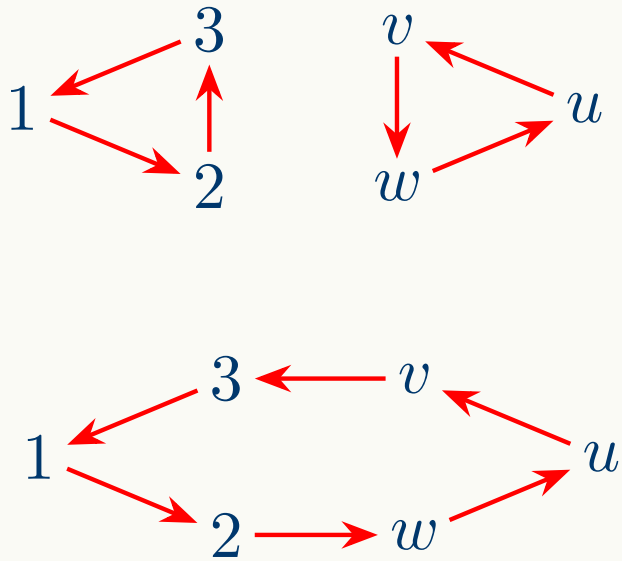

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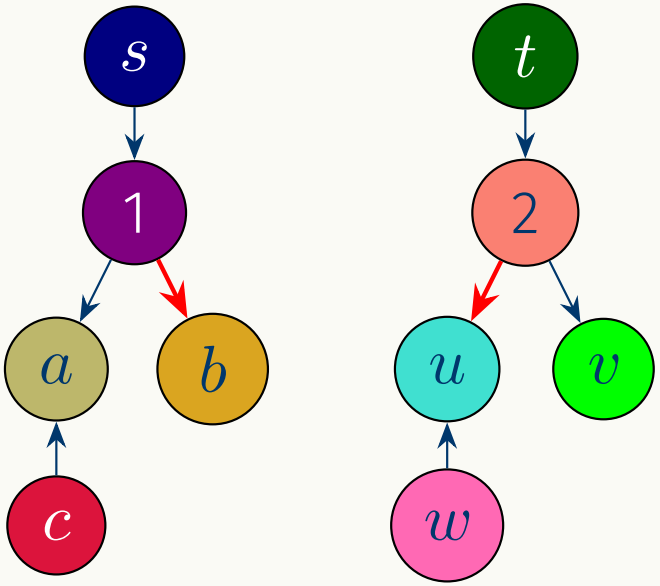

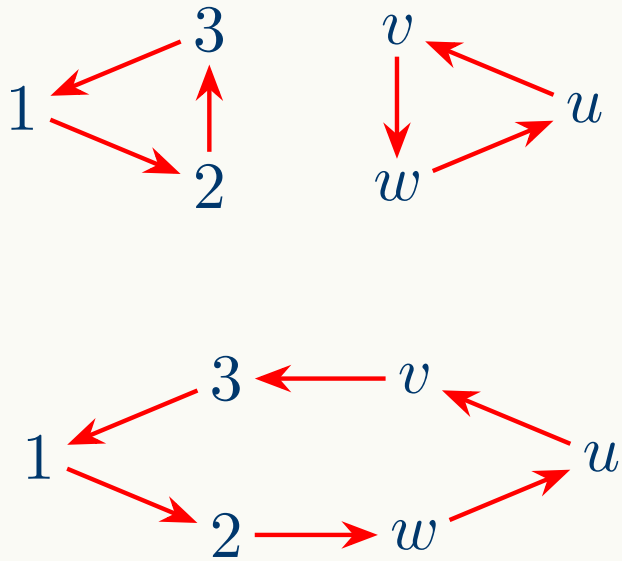

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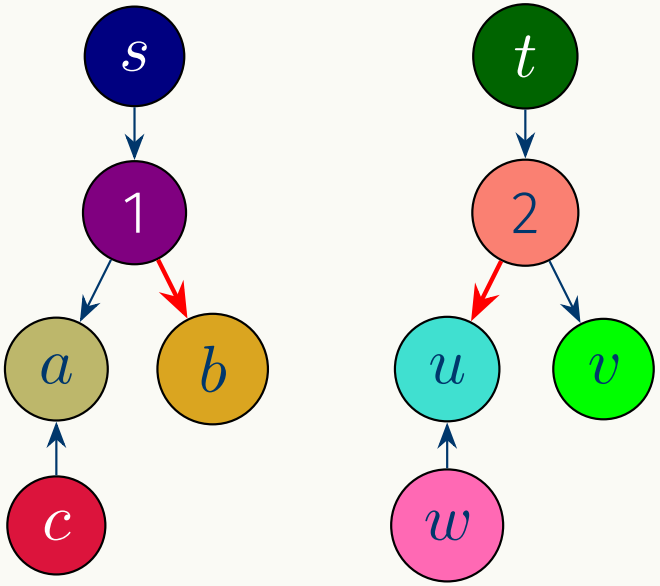

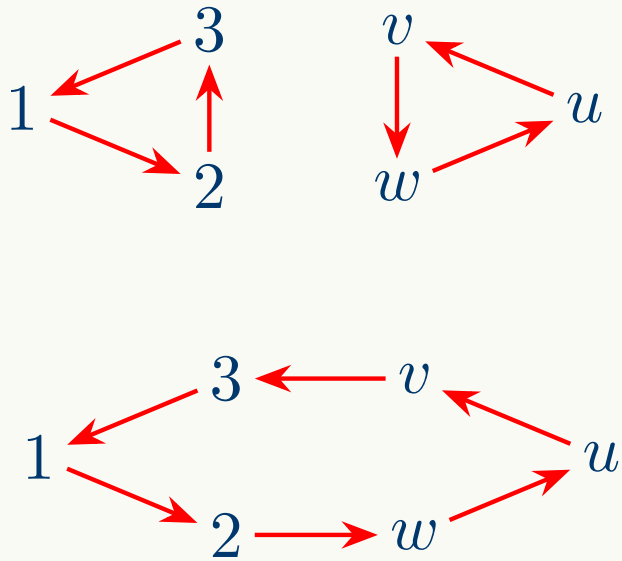

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Wanted: similar method for more complex objects.

In this talk: Relational structures.

But how to identify the “right” approach?

Key Properties of Color Refinement

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And RCR should run in time $\mathcal{O}\big(\|\mathcal{A}\| \cdot \log \|\mathcal{A}\|\big)$.

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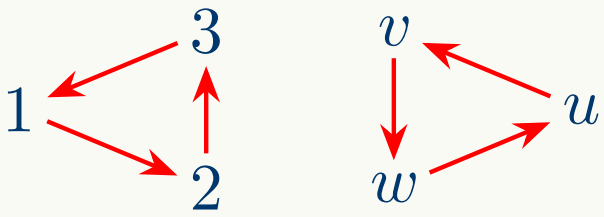

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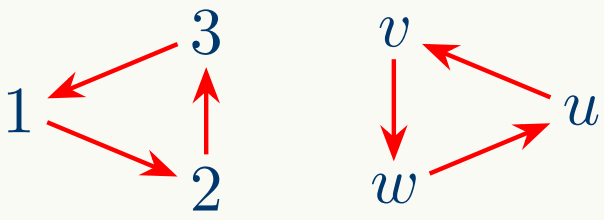

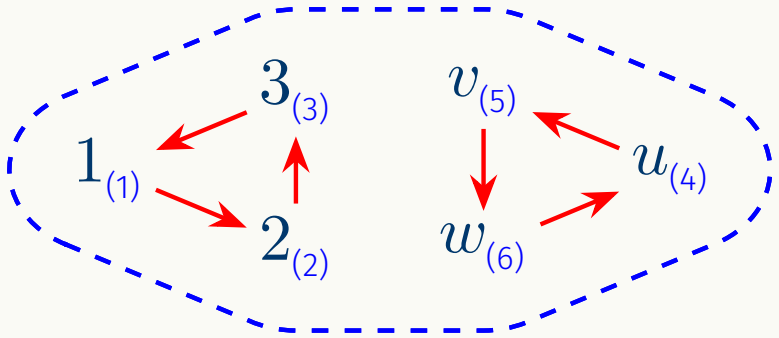

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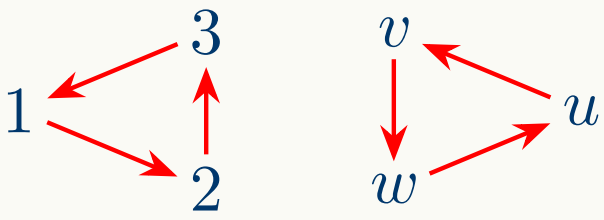

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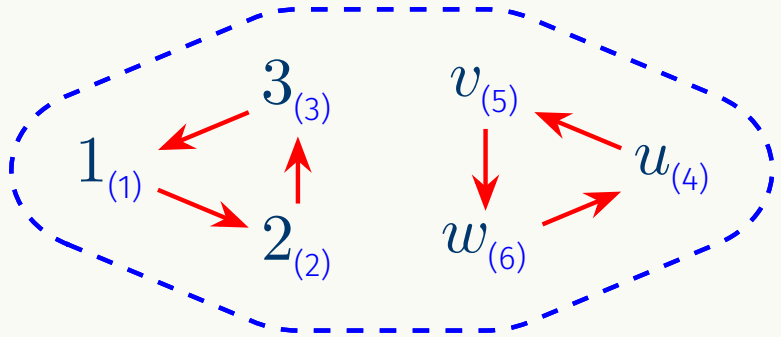

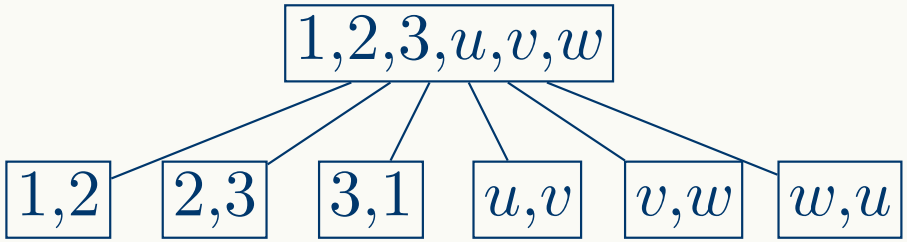

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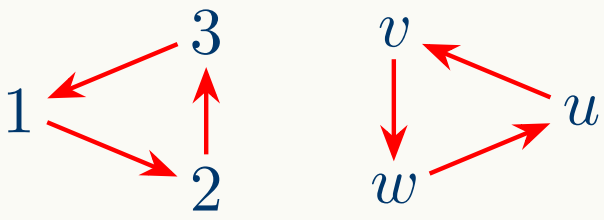

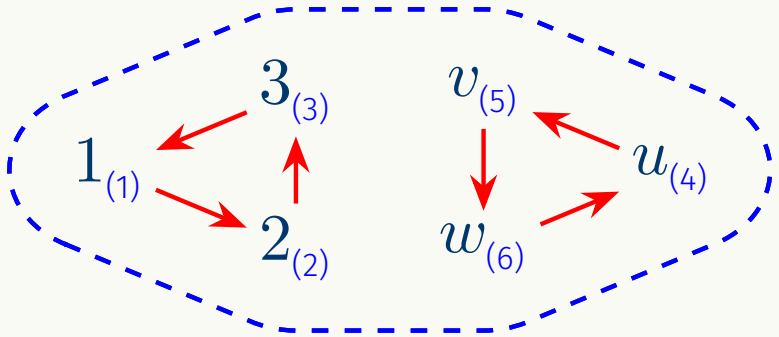

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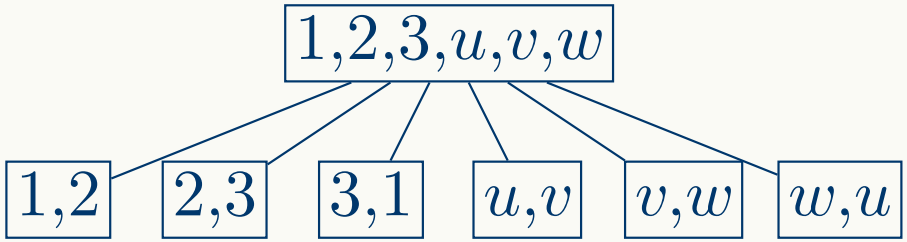

- Standard, most general notion from DB theory, (Abiteboul, Hull, Vianu 1995).

The Guarded Fragment of FO with Counting Quantifiers

Atomic formulas and Boolean operators as usual:

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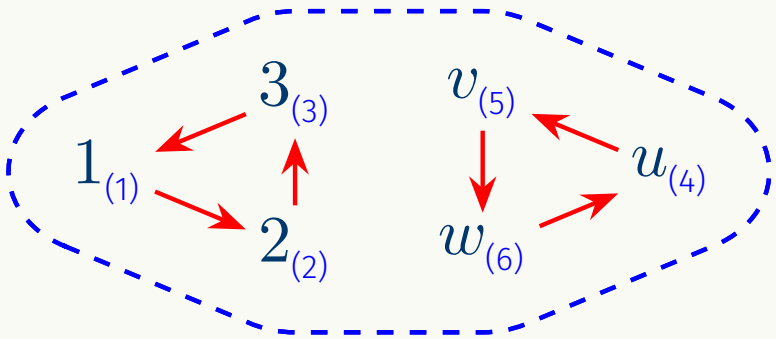
GF(C) — Example

$R^{\mathcal{A}}$	$1,2,3,u,v,w$		
$E^{\mathcal{A}}$	$1,2$	$2,3$	$3,1$
	u,v	v,w	w,u



$R^{\mathcal{B}}$	$1,2,3,u,v,w$		
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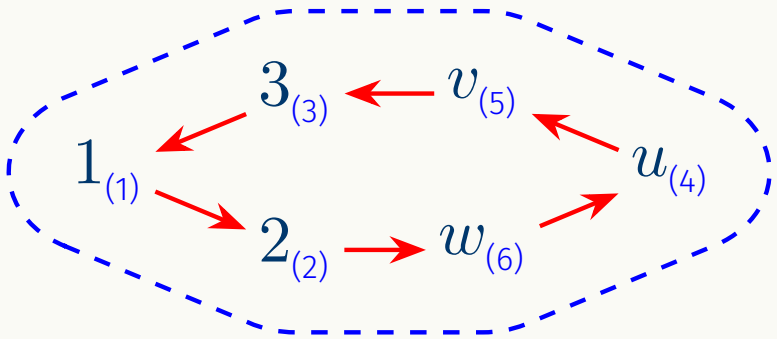
	u,v	v,w	w,u



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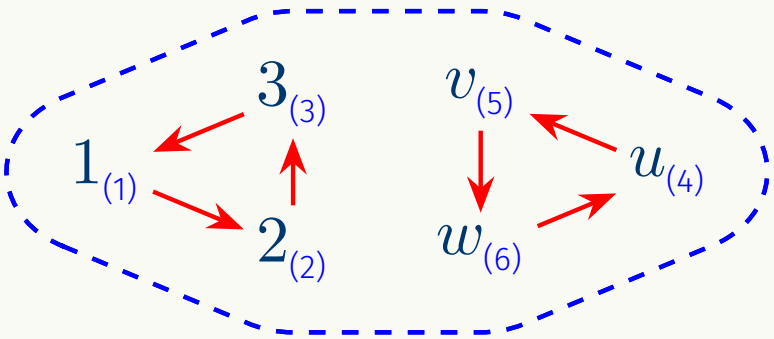

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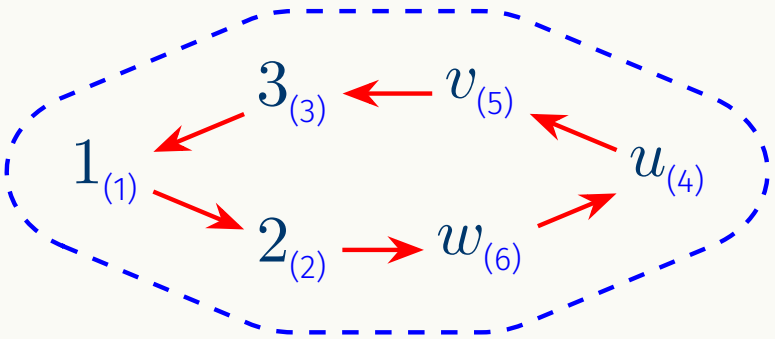

$$\hom(\mathcal{A},\mathcal{B}) = 0 \quad \text{and} \quad \hom(\mathcal{A},\mathcal{A}) > 0$$

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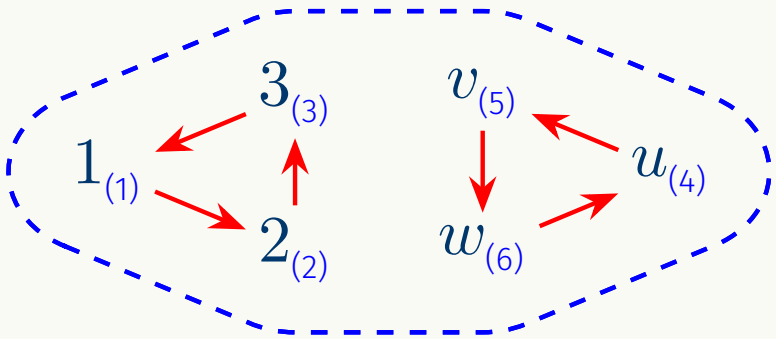

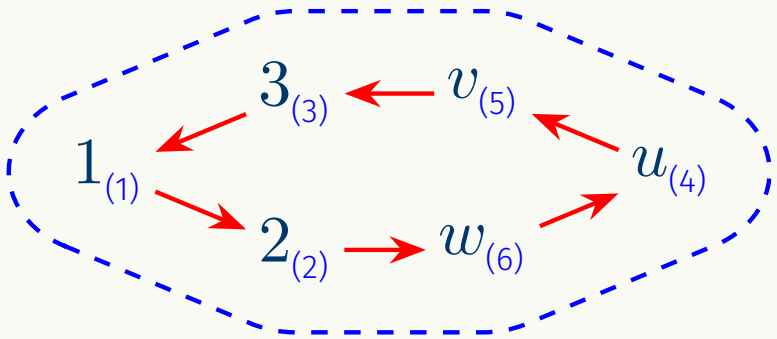

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$$\exists^{\geq 1}(x_1,\dots,x_6).\Big(R(x_1,\dots,x_6) \wedge \bigwedge_{1\leq i\leq 5} E(x_i,x_{i+1}) \wedge E(x_6,x_1)\Big)$$

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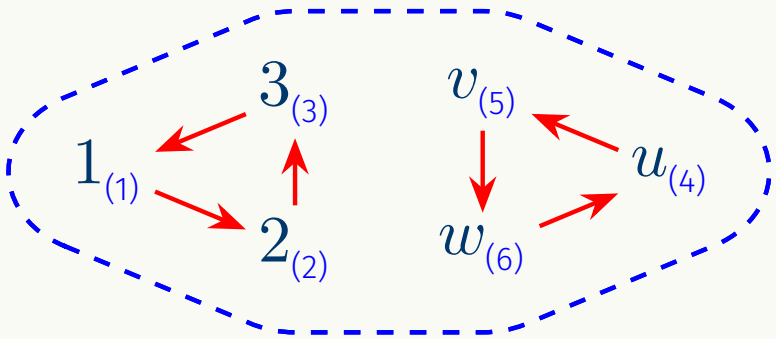

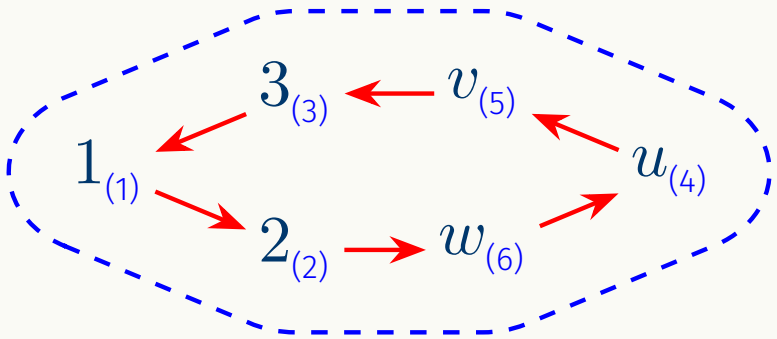

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RCR should also distinguish $\mathcal{A}$ and $\mathcal{B}$!

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Idea: Color *tuples* instead of *elements*:

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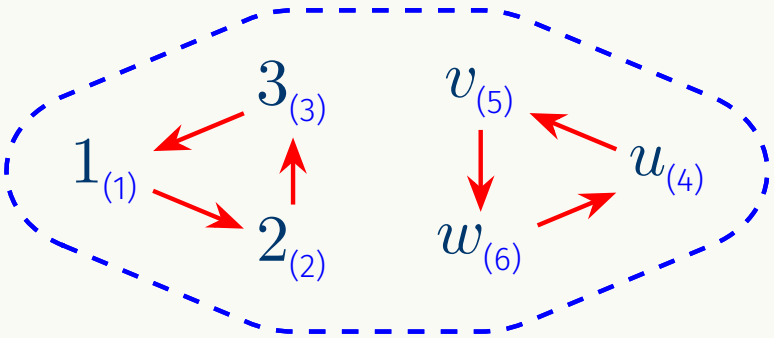

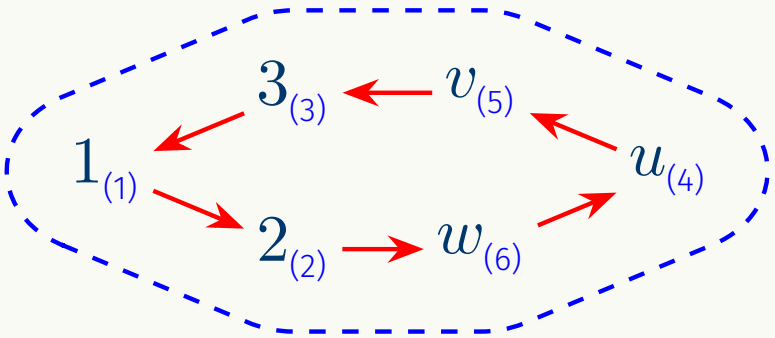

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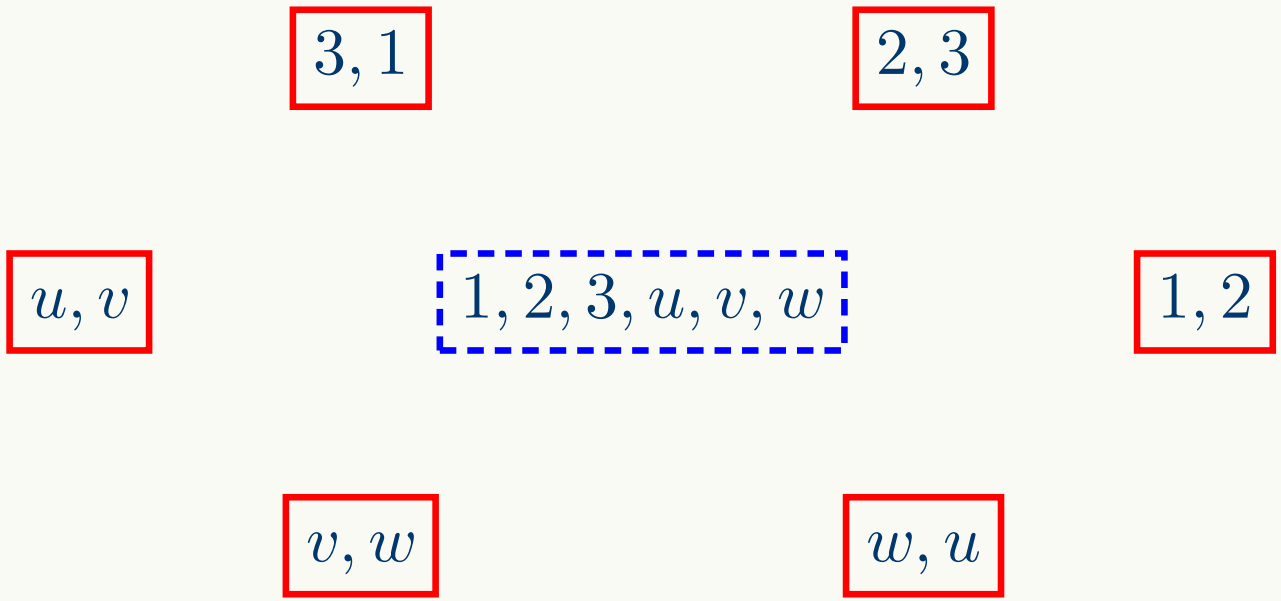

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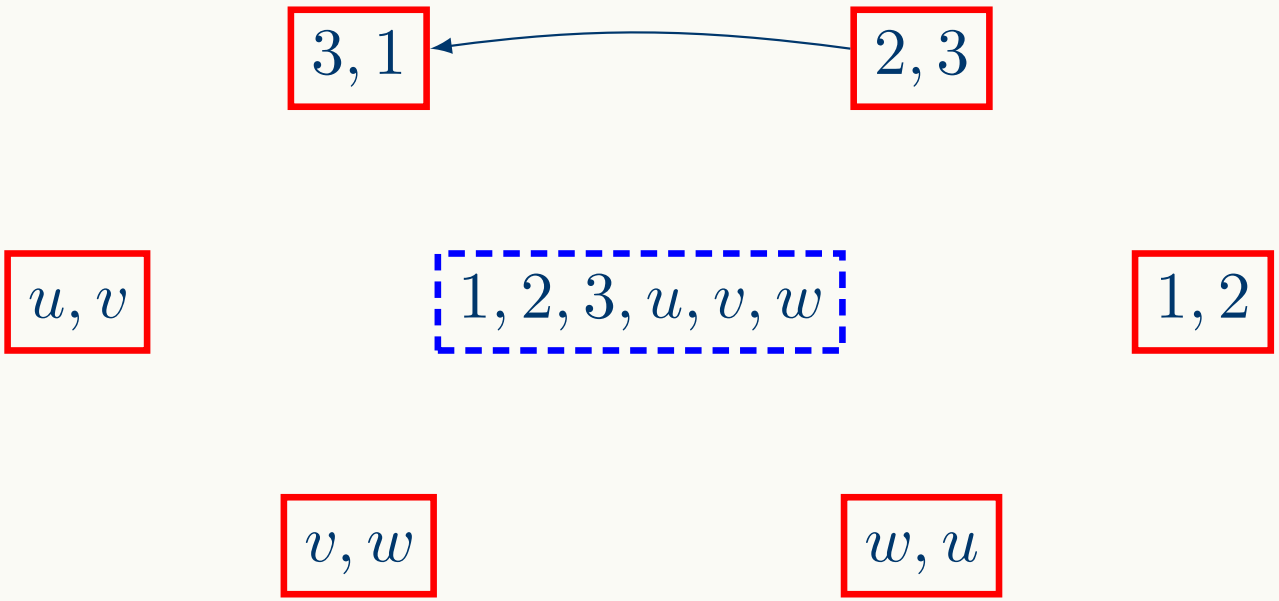

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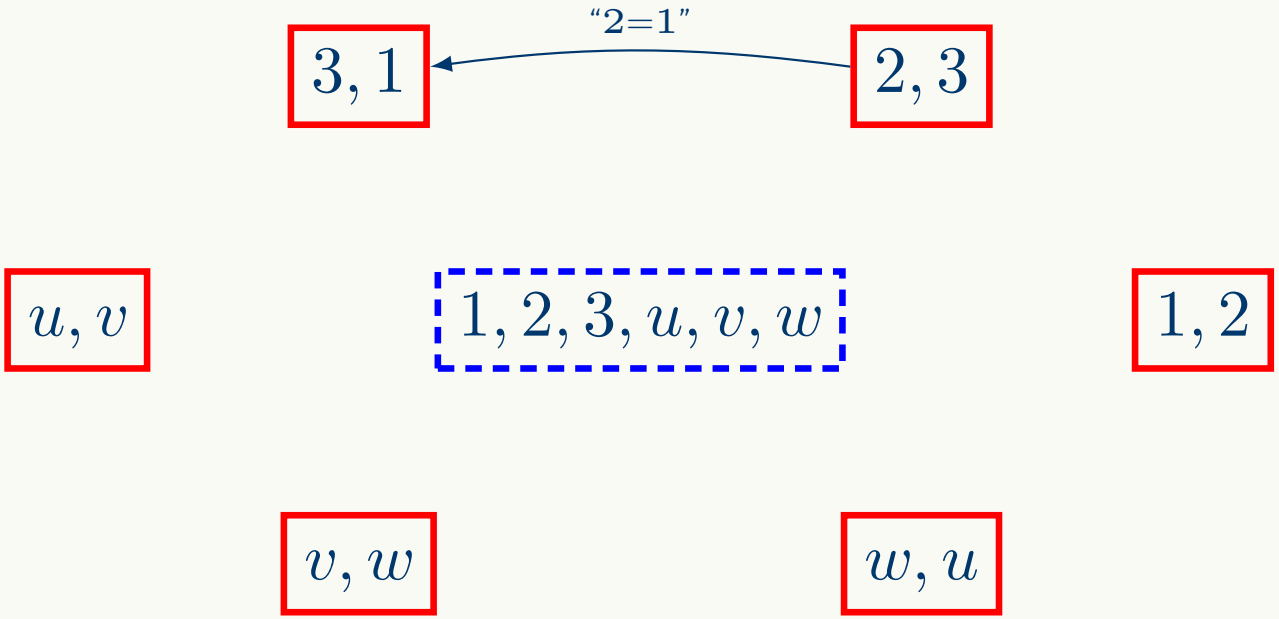

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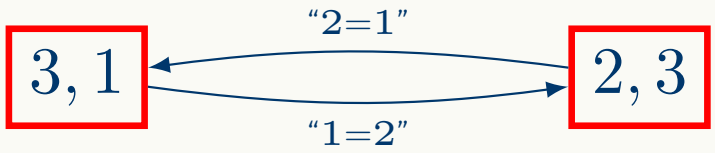

u, v

$1, 2, 3, u, v, w$

$1, 2$

v, w

w, u

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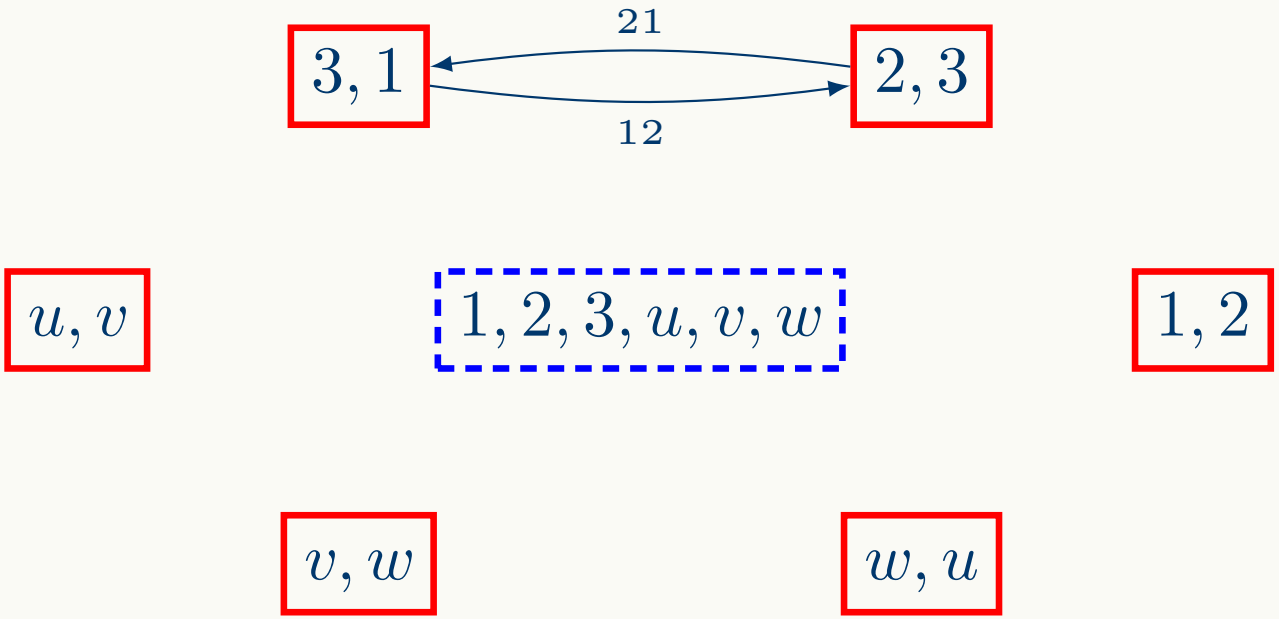

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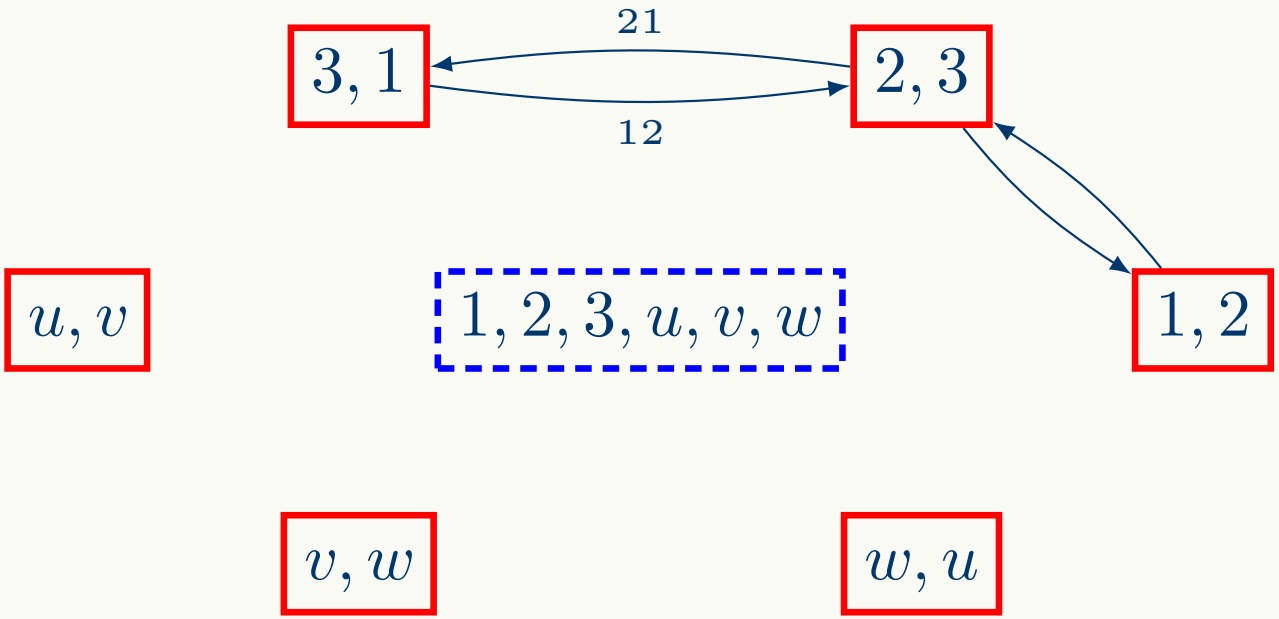

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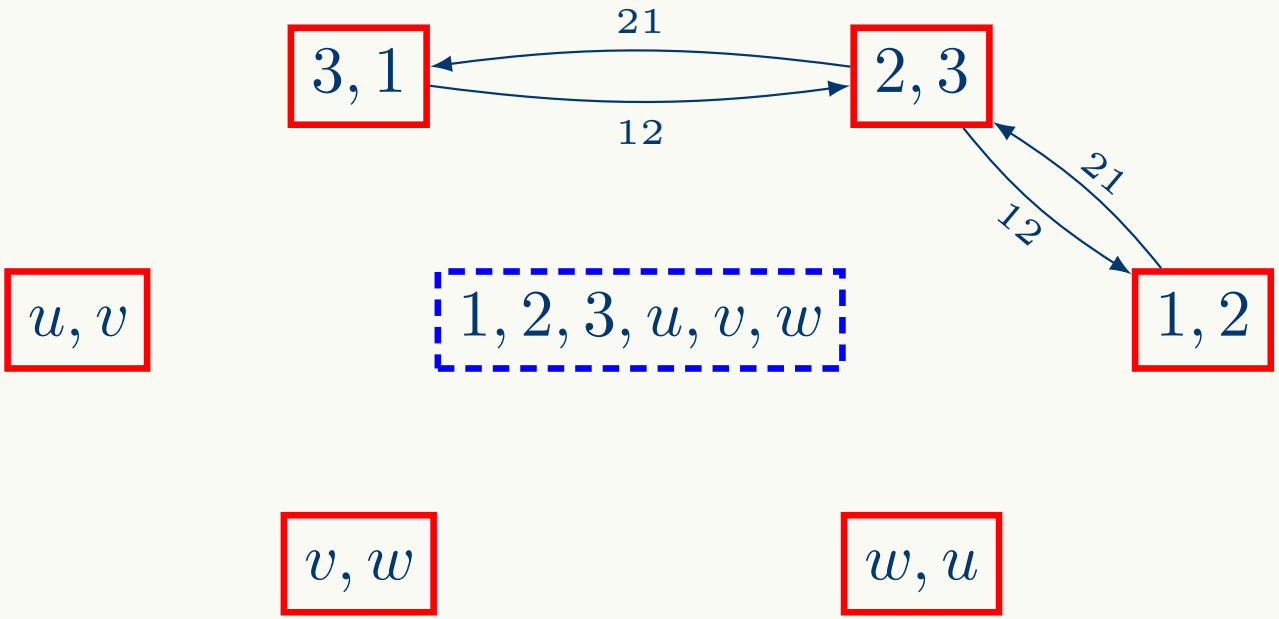

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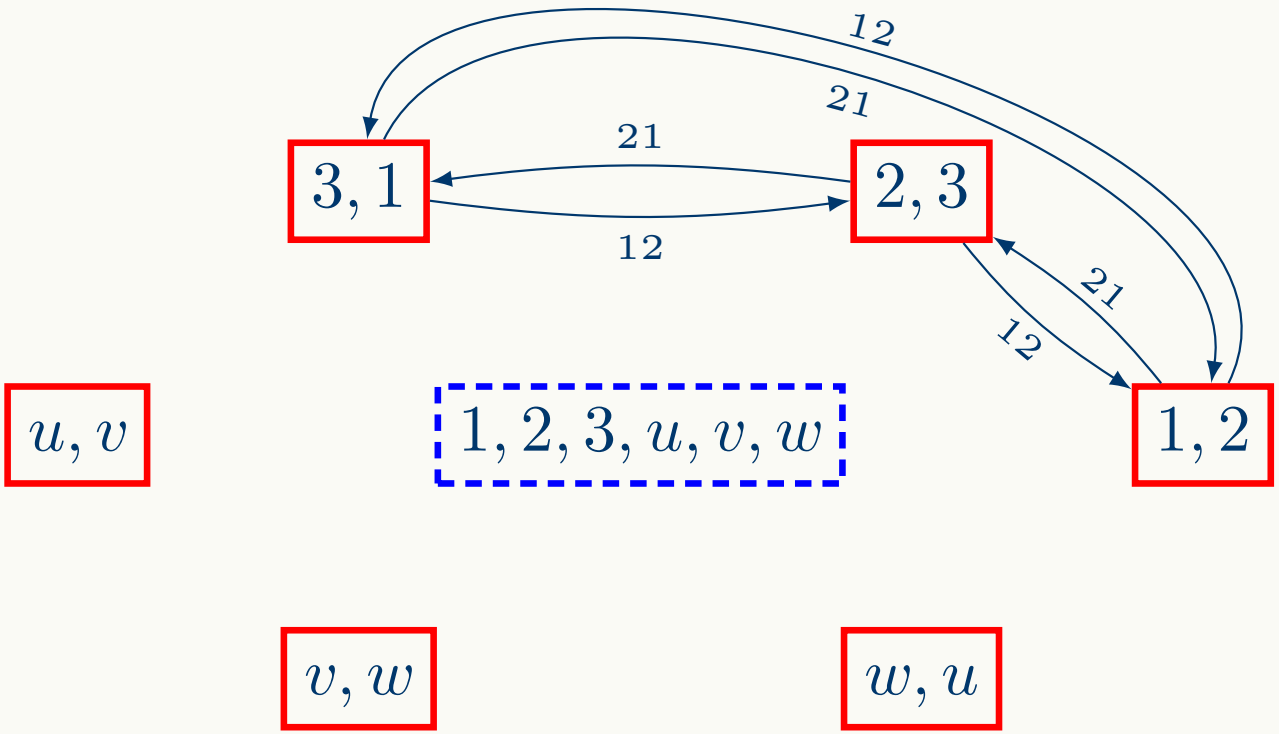

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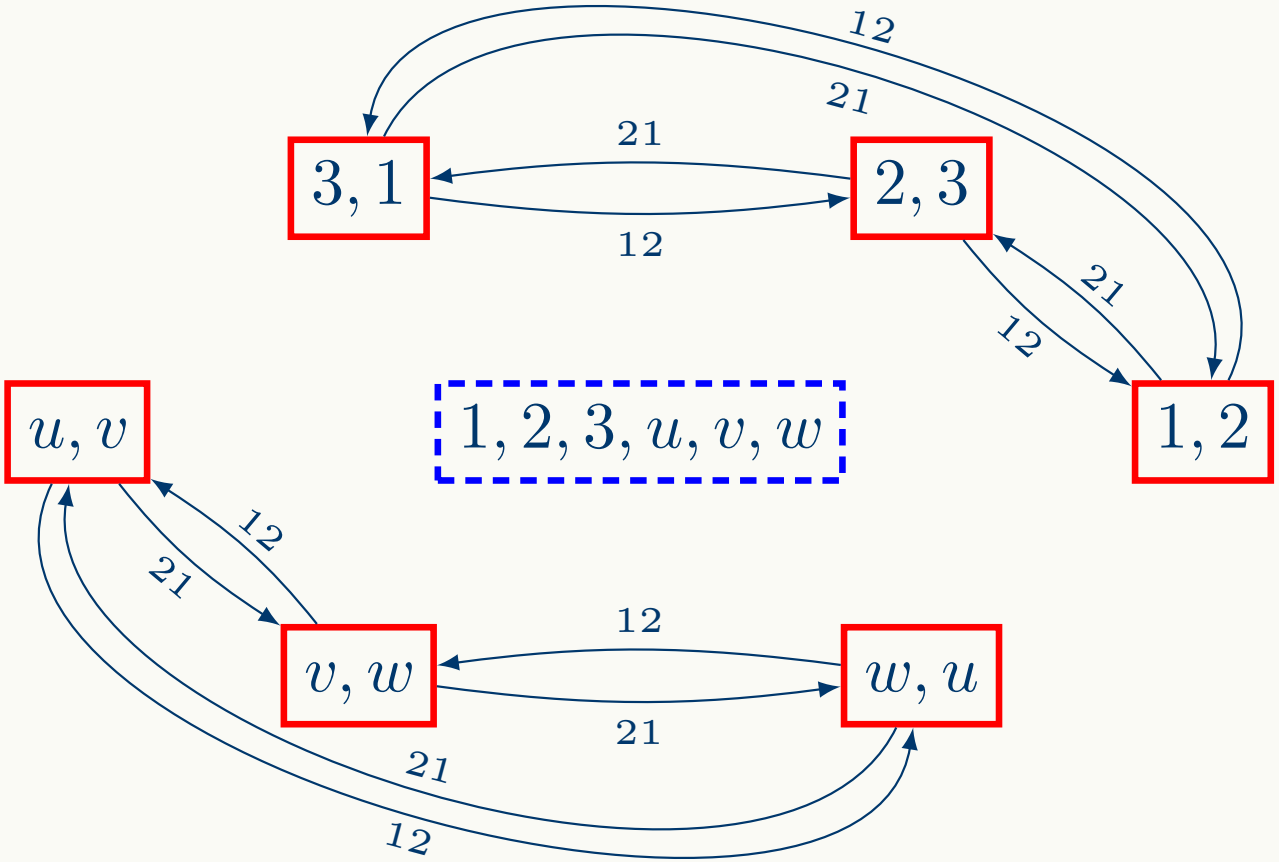

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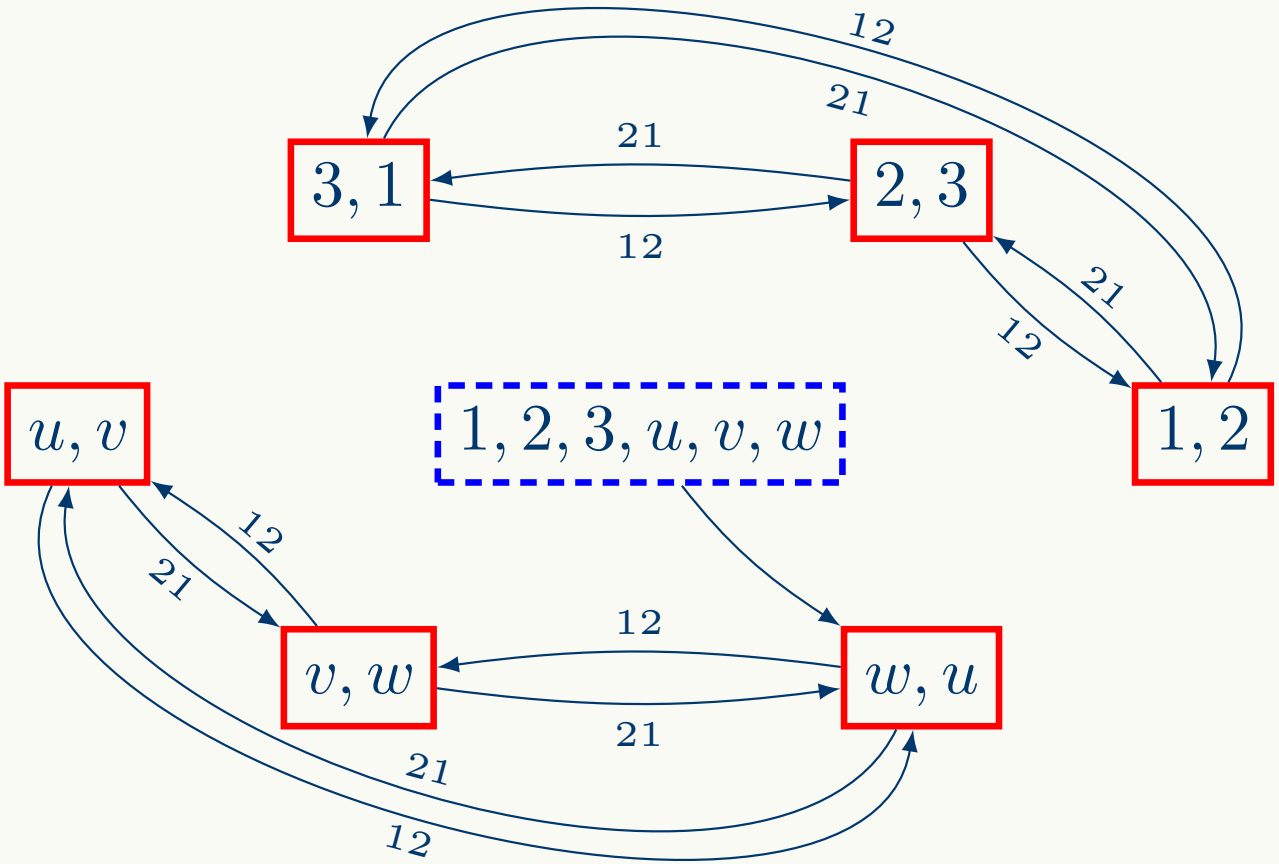

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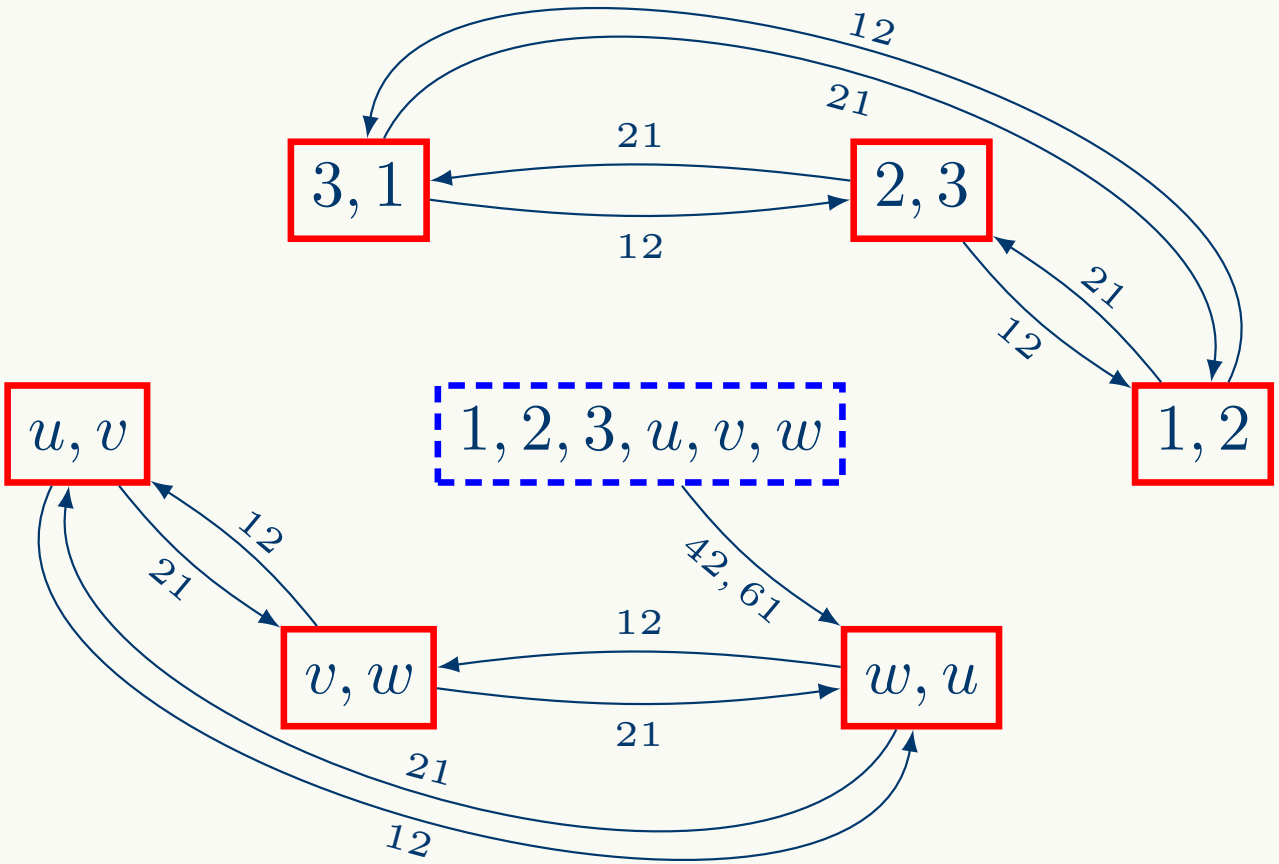

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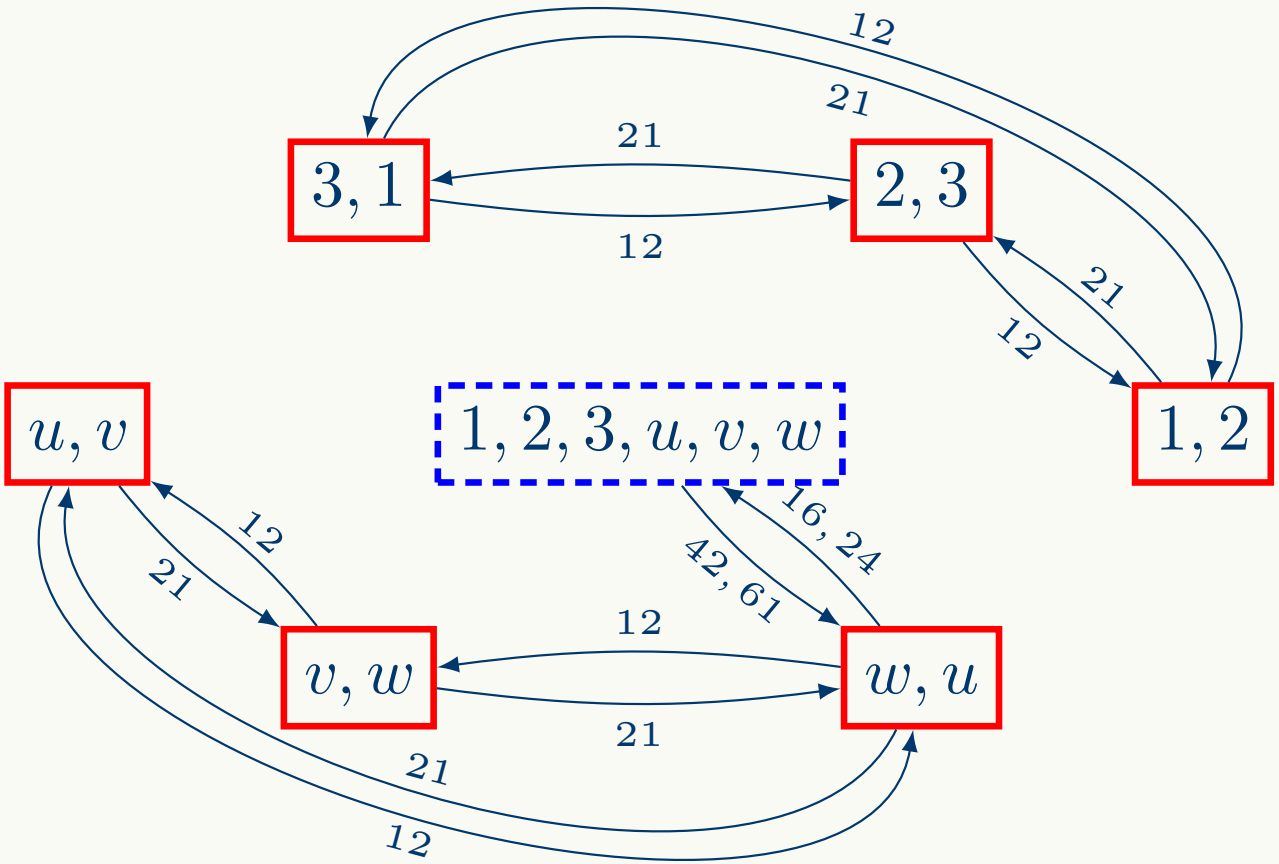

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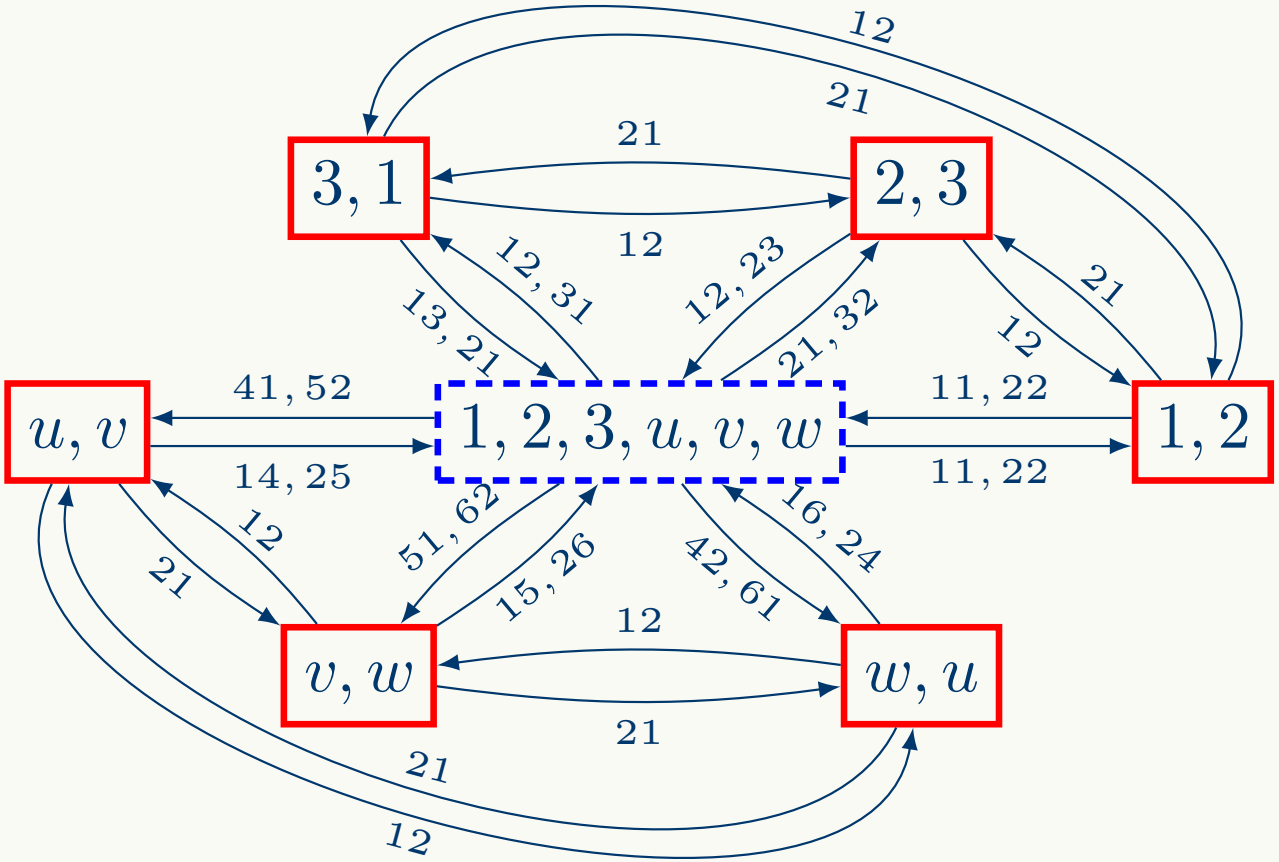

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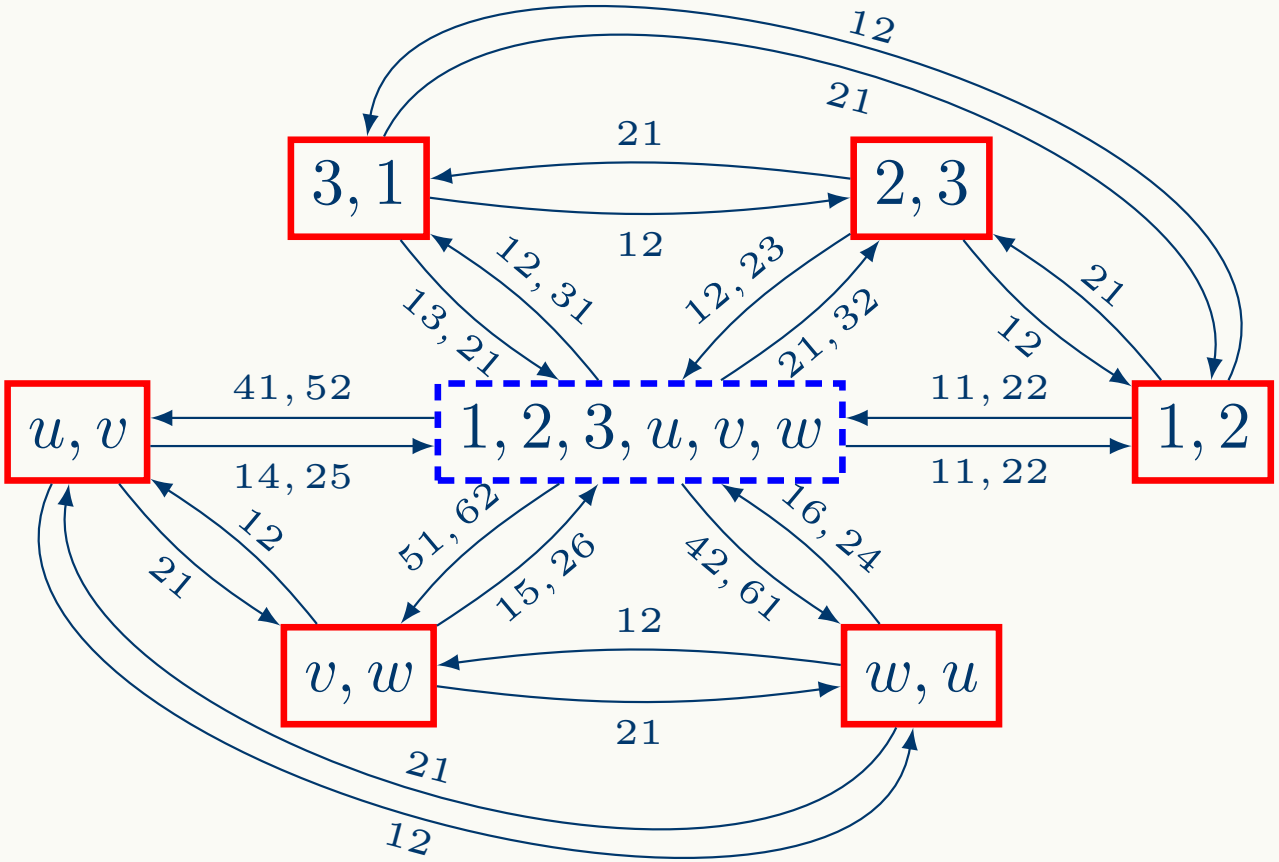

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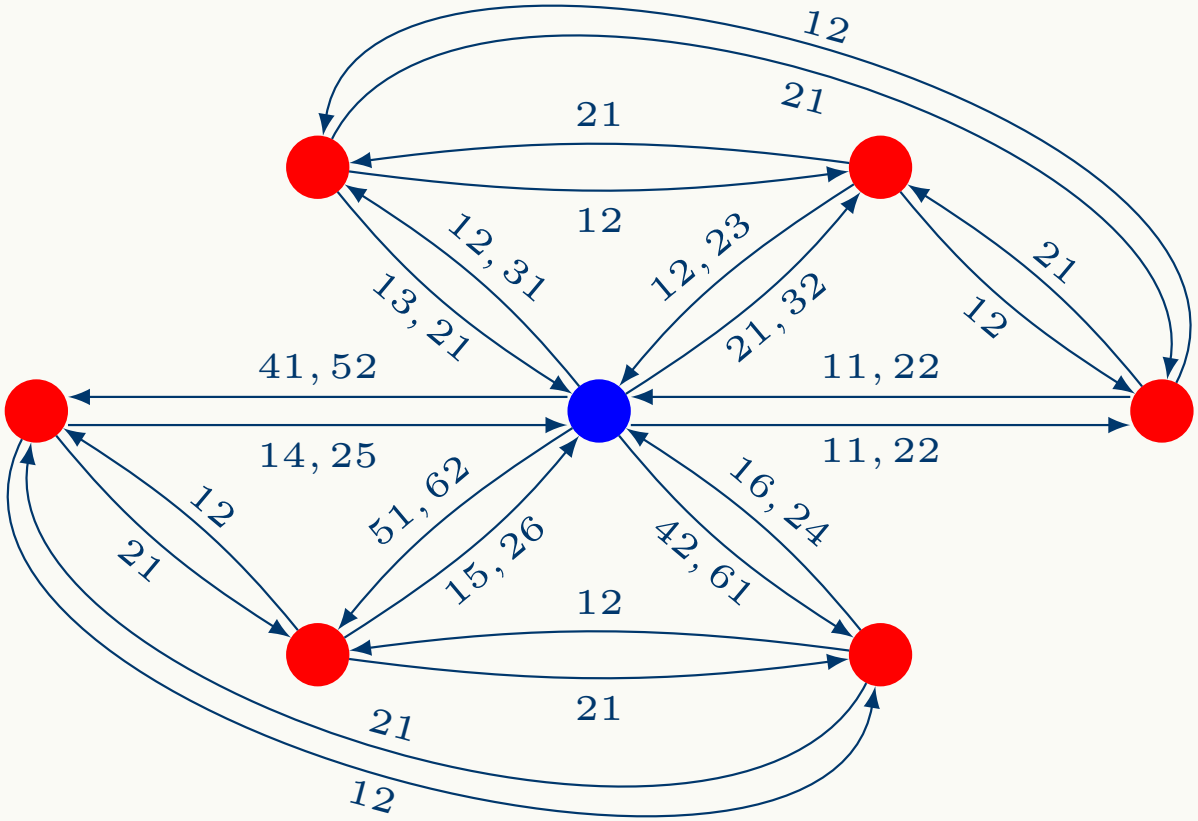

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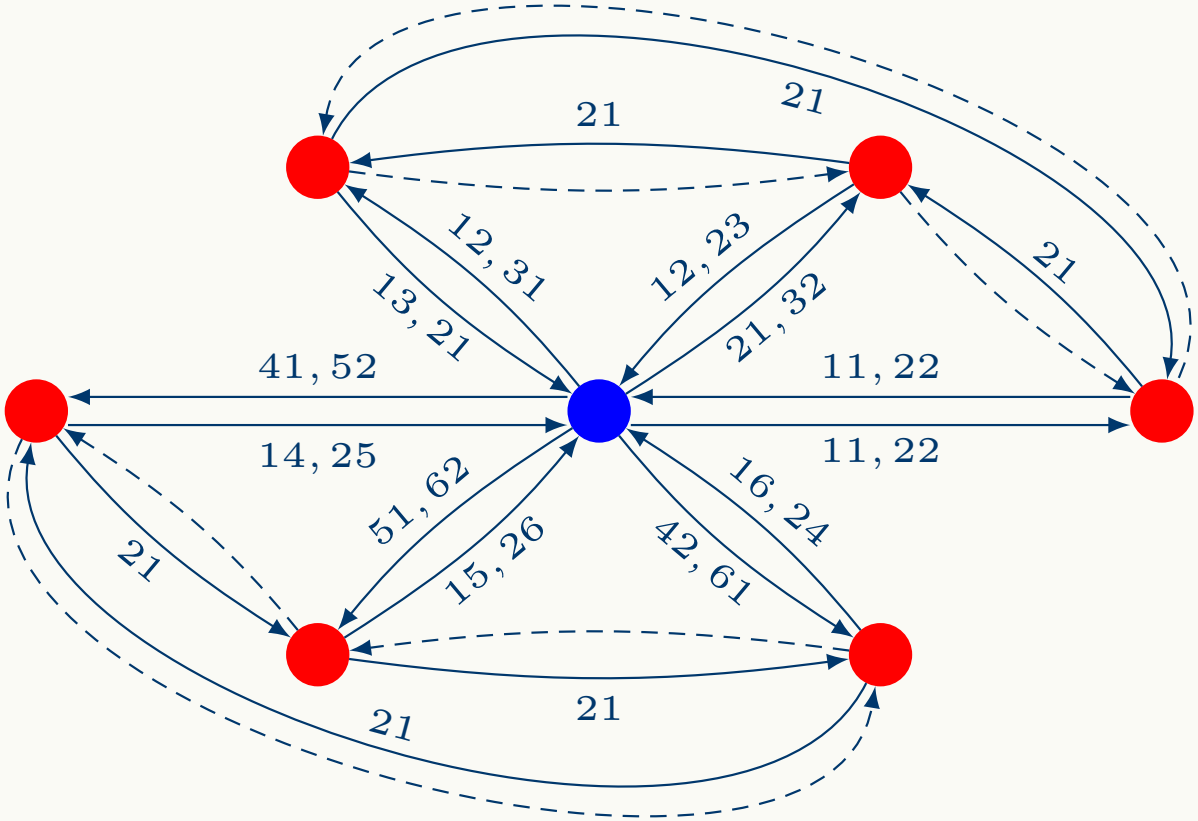

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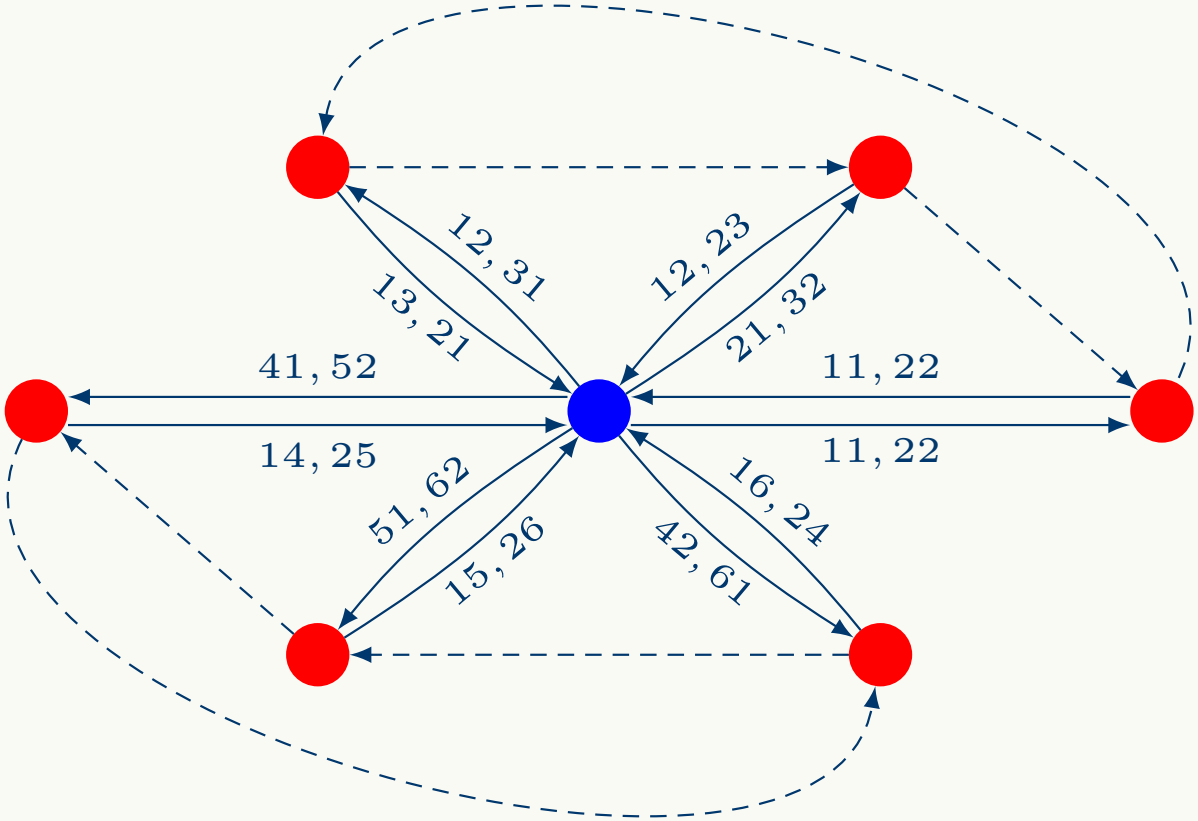

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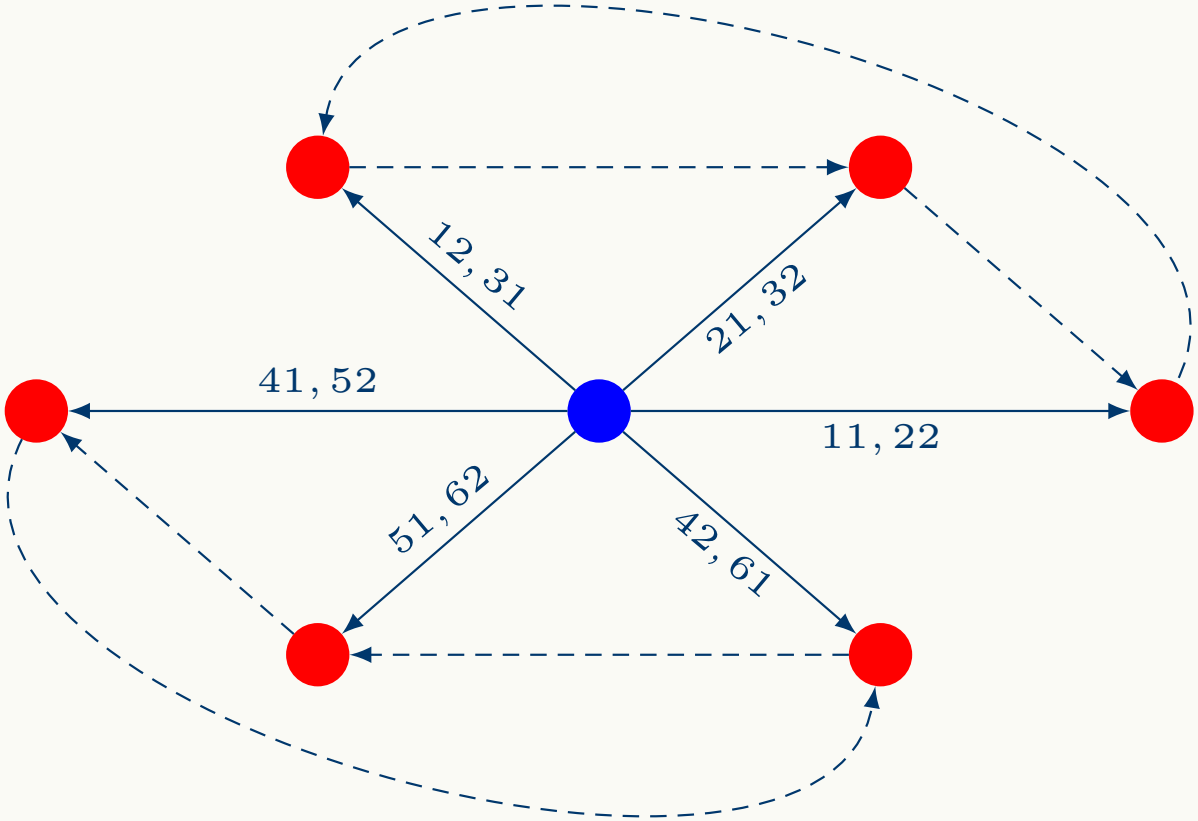
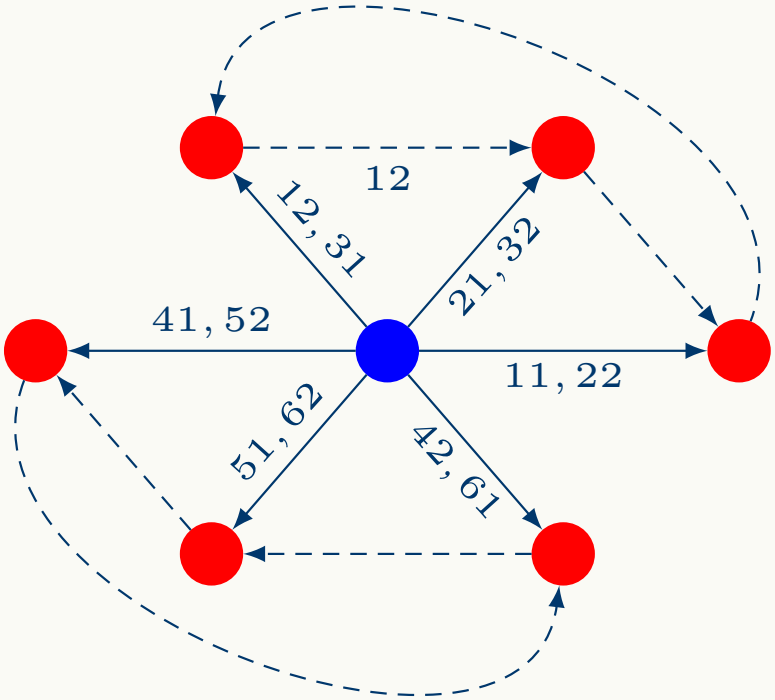
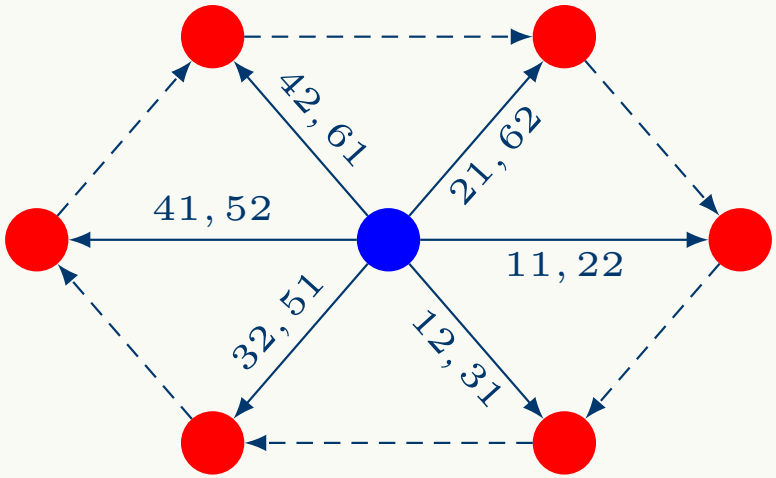


RCR on $\mathcal{A}, \mathcal{B}$ is equivalent to CR on $G_{\mathcal{A}}, G_{\mathcal{B}}$





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- So RCR runs in time $\mathcal{O}(\|\mathcal{A}\|^2 \cdot \log \|\mathcal{A}\|)$ in the worst case?
- Didn't I promise $\mathcal{O}(\|\mathcal{A}\| \cdot \log \|\mathcal{A}\|)$?!

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Use a different encoding $H_{\mathcal{A}}$ of $\mathcal{A}$ instead of $G_{\mathcal{A}}$ where

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- The proof that CR on $H_{\mathcal{A}}$ is equivalent to RCR on $\mathcal{A}$ is quite involved.

Isn’t this way too complicated?!

Let us look at “straightforward” approaches.

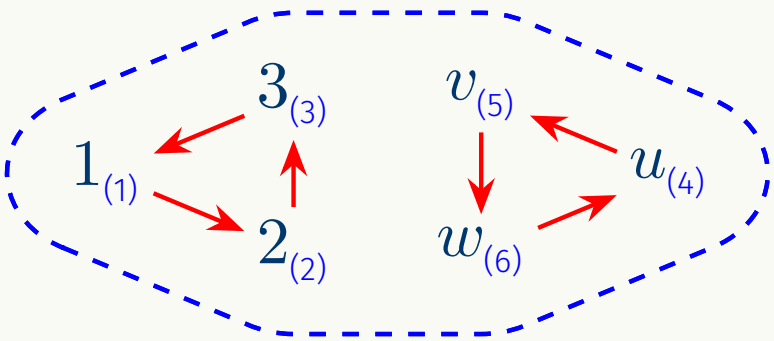
Approach 1:

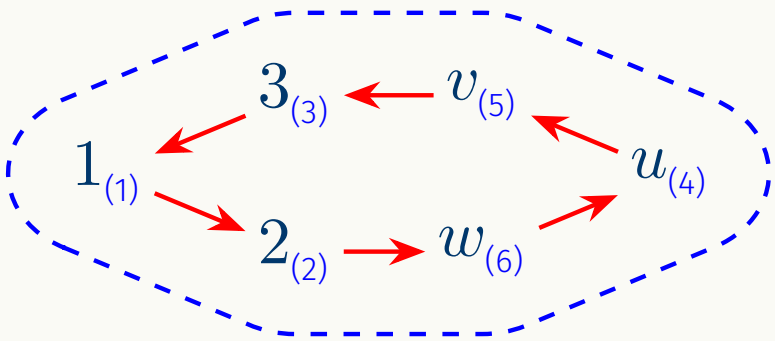
Run Color Refinement on the Gaifman graph.

Approach 2:

Run Color Refinement on the incidence graph.

Consider our Examples...





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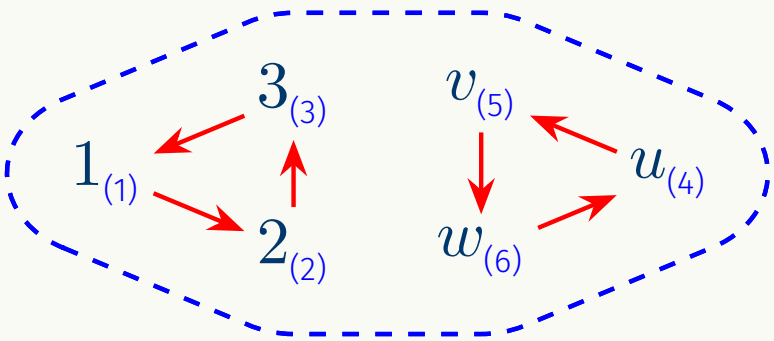
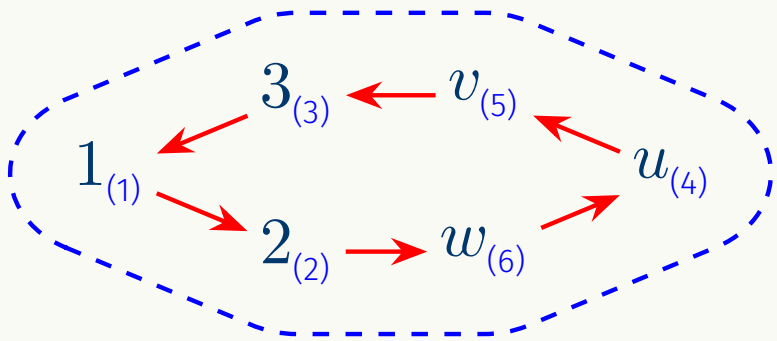
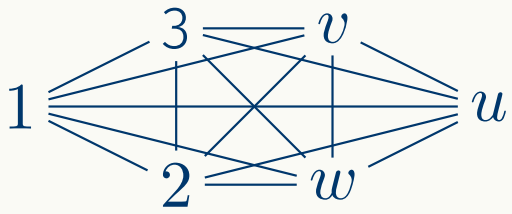
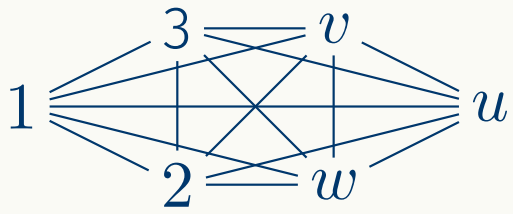
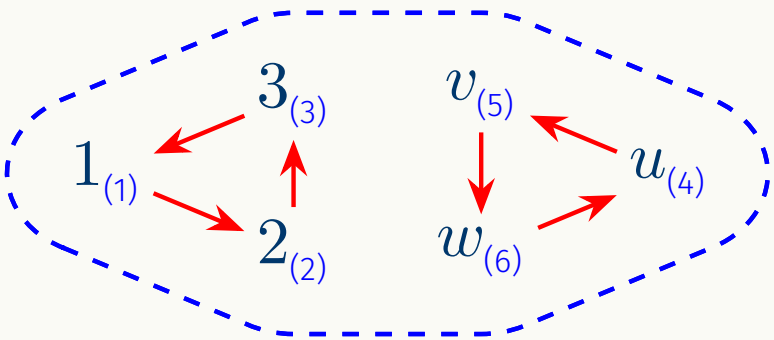


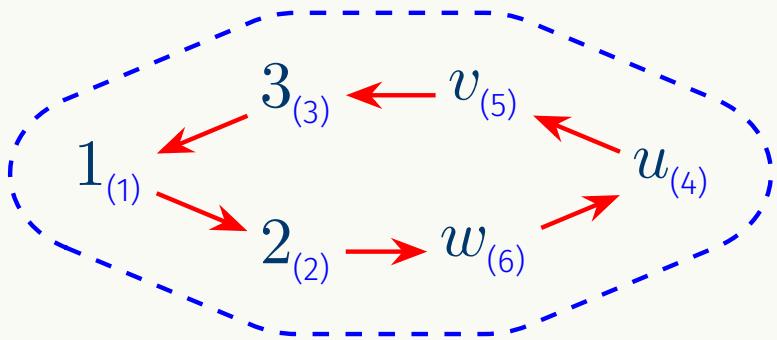


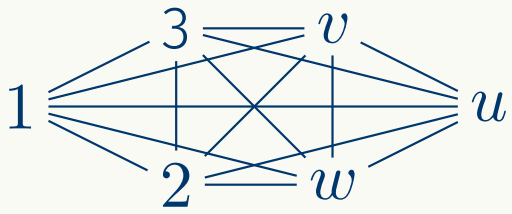


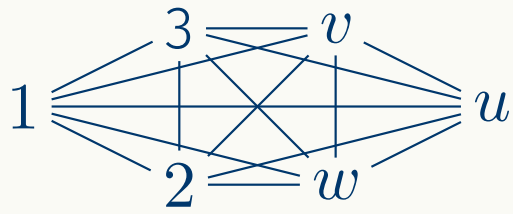


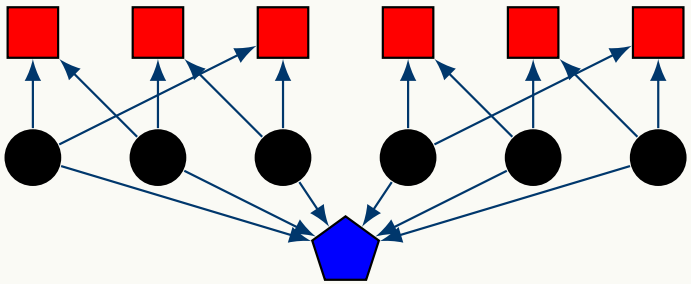
Consider our Examples...

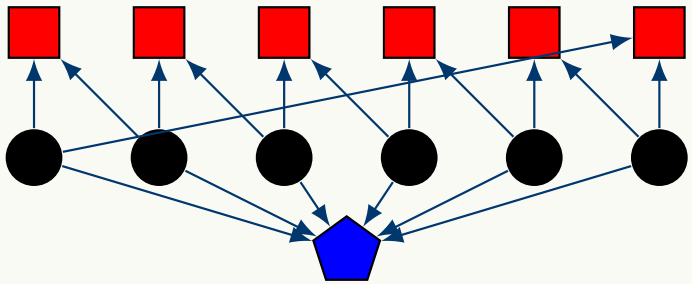




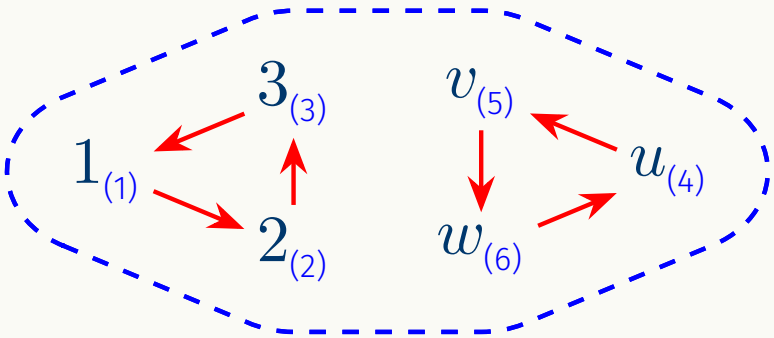


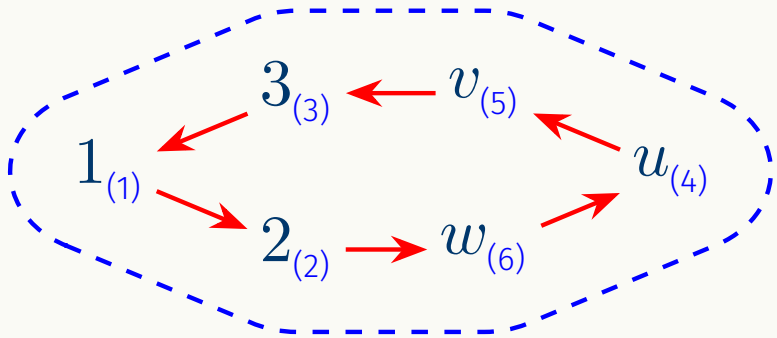


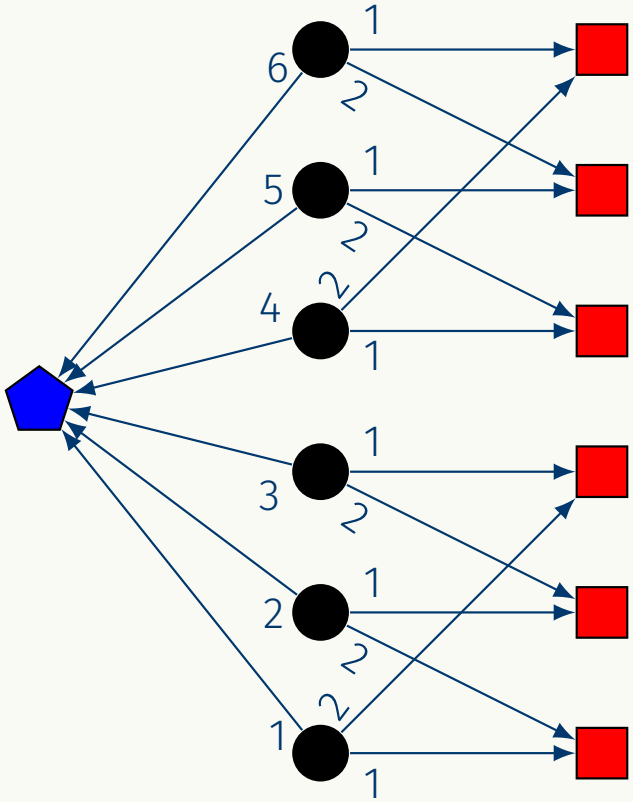


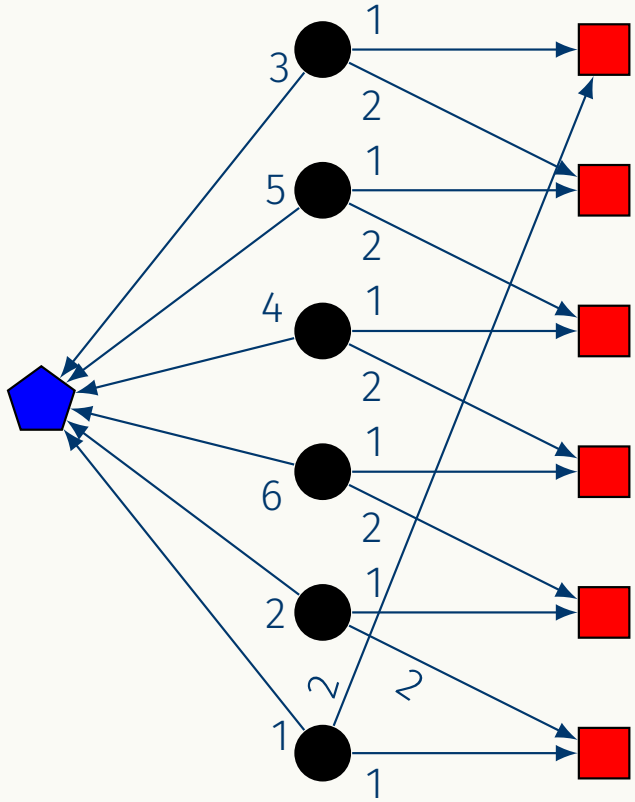


Press more Information into the Representations

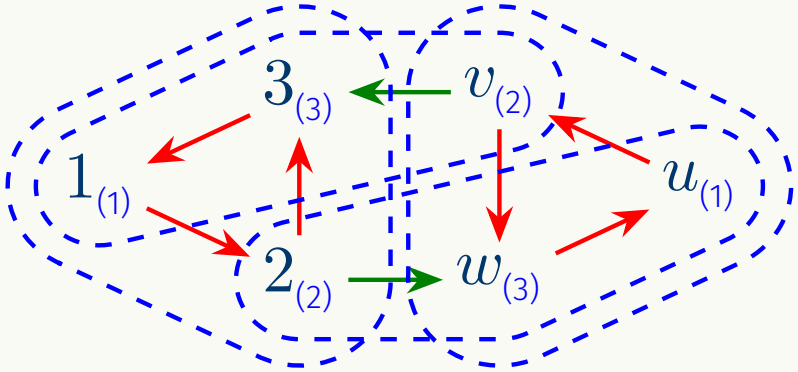


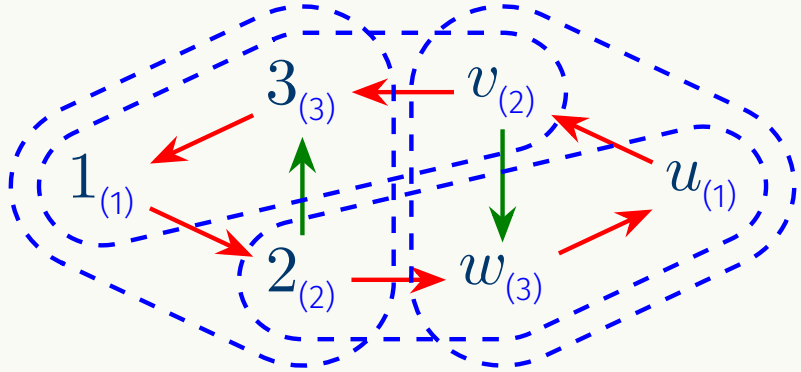




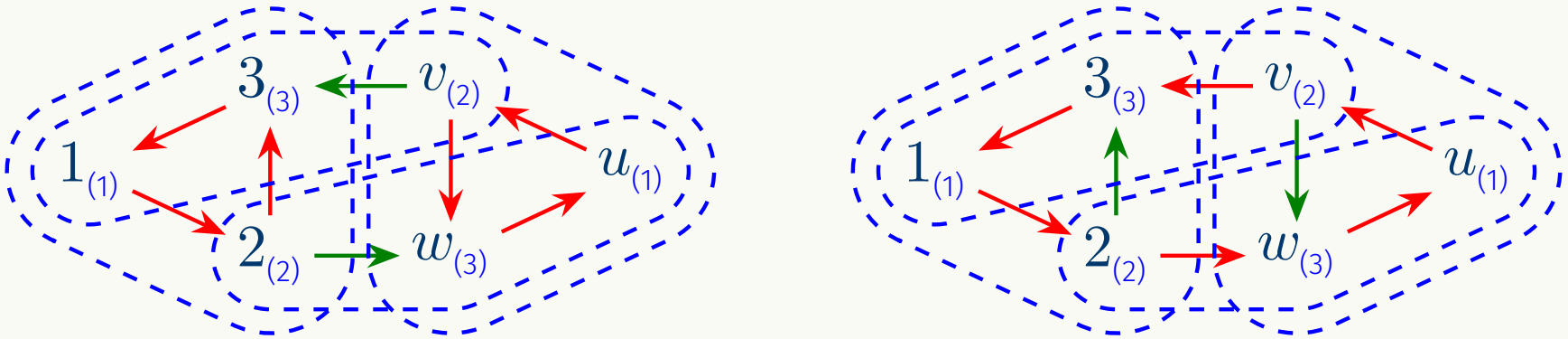


Another Counter-Example



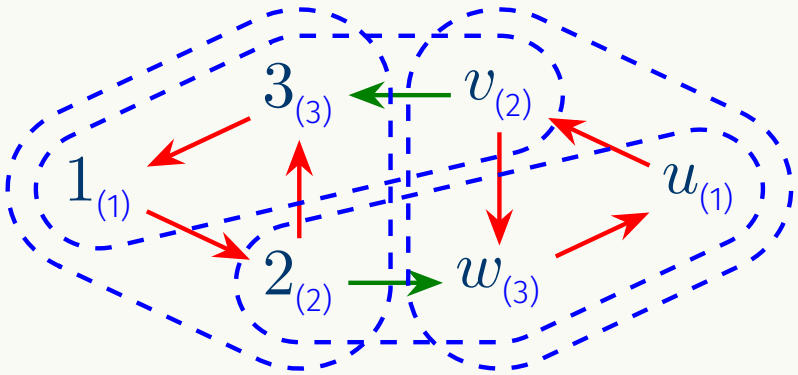


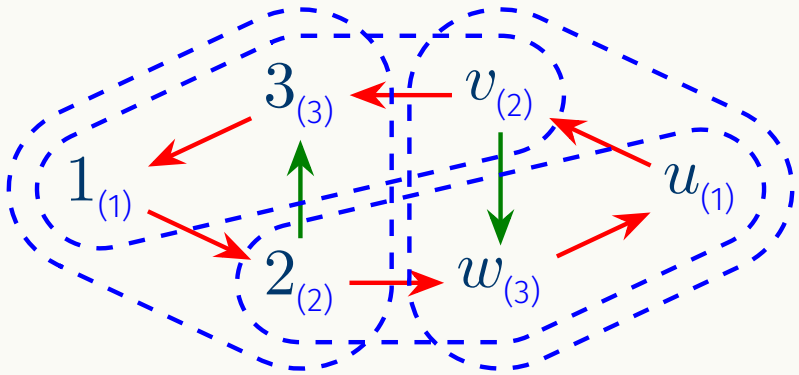
Another Counter-Example



CR does not distinguish their incidence graphs, even with labeled edges.

Another Counter-Example





CR does not distinguish their incidence graphs, even with labeled edges.

$$\exists^{\geq 1}(x_1, x_2, x_3)\,.\,\big(\,R(x_1, x_2, x_3) \wedge E(x_1, x_2) \;\wedge\; F(x_2, x_3) \wedge E(x_3, x_1)\,\big)$$

Bottom line:

These representations do not seem to capture homomorphisms from acyclic structures.

They somehow lose crucial information.

Main Result & Outlook

Theorem (Scheidt, Schweikardt 2024)

The following are equivalent:

- *Relational Color Refinement distinguishes $\mathcal{A}$ and $\mathcal{B}$.*
- *There is an acyclic $\mathcal{C}$ s.t. $\hom(\mathcal{C}, \mathcal{A}) \neq \hom(\mathcal{C}, \mathcal{B})$.*
- *There is a $\varphi \in \mathrm{GF}(\mathrm{C})$ s.t. $\mathcal{A} \models \varphi$, and $\mathcal{B} \not\models \varphi$.*
- *Spoiler has a win. strategy for the Guarded-Game on $\mathcal{A}$, $\mathcal{B}$.*

RCR can be implemented to run in time $\mathcal{O}(\|\mathcal{A}\| \cdot \log \|\mathcal{A}\|)$.

Outlook: lift results on graphs to relational structures?

- CR can be used to speed up the evaluation of fc-ACQs on binary schemas (Riveros, Scheidt, Schweikardt 2024).
- CR is related to Graph Neural Networks.
- k-dimensional Relational Weisfeiler-Leman?

Further Reading

📝 Benjamin Scheidt, Nicole Schweikardt.
“Color Refinement for Relational Structures”.
2024. DOI: 10.48550/ARXIV.2407.16022.

📝 Benjamin Scheidt, Nicole Schweikardt.
“Counting Homomorphisms from Hypergraphs of Bounded Generalised Hypertree Width: A Logical Characterisation”.
2023. DOI: 10.4230/LIPIcs.MFCS.2023.79.

📝 Zdeněk Dvořák.
“On recognizing graphs by numbers of homomorphisms”.
2010. DOI: 10.1002/jgt.20461.

Other References

📝 Cristian Riveros, Benjamin Scheidt, Nicole Schweikardt.
“Using Color Refinement to Boost Enumeration and Counting for Acyclic CQs of Binary Schemas”.
2024.

📝 Holger Dell, Martin Grohe, Gaurav Rattan.
“Lovász Meets Weisfeiler and Leman”.
2018. DOI: 10.4230/LIPIcs.ICALP.2018.40.

📝 Serge Abiteboul, Richard Hull, Victor Vianu.
Foundations of Databases.
1995.

📝 Neil Immerman, Eric Lander.
“Describing Graphs: A First-Order Approach to Graph Canonization”.
1990. DOI: 10.1007/978-1-4612-4478-3_5.