HOMOTOPY TYPES AS HOMOTOPY TYPES

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Abstract. Quillen famously proved that the homotopy theory of spaces can be modeled by the category of simplicial sets, with Kan complexes encoding homotopy types. Voevodsky extended this result to show that homotopy type theory — Martin-Löf's dependent type theory plus the univalence axiom — can be modelled by the category of simplicial sets, again with the Kan complexes encoding homotopy types. In this talk, I will explain what features must be added to Quillen's model structure to obtain a model of homotopy type theory and then explain why a Quillen equivalent model, on a suitably chosen category of cubical sets, may be preferred. This last part involves joint work with Steve Awodey, Evan Cavallo, Thierry Coquand, and Christian Sattler.

Contents

1. Simplicial sets as homotopy types

Following Eilenberg and Zilber [EZ], Gabriel and Zisman [GZ], Kan [K], Quillen [Q] and others, the classical homotopy theory of homotopy types can be modeled by the category of simplicial sets $s\mathcal{S}et := \mathcal{S}et^{\mathbb{\Delta}^{op}}$.

(1.0.1)
$$\begin{tikzcd} & \mathbb{\Delta} \arrow[dl, hook', "よ"'] \arrow[dr, "\Delta^\bullet"] & \\ s\mathcal{S}et \arrow[rr, bend left, "|-|:=\mathrm{colim}_{\Delta^\bullet}"] \arrow[rr, phantom, "\perp"] & & \mathcal{T}op \arrow[ll, bend left, "\mathrm{Sing}:=\hom(\Delta^\bullet,-)"] \end{tikzcd}$$

Theorem 1.0.2 (Quillen). *There is a right proper simplicial model structure on simplicial sets whose cofibrations are the monomorphisms.*

Moreover, the model structure of Theorem 1.0.2 recovers classical homotopy theory, in the sense that (1.0.1) is a Quillen equivalence.

Consequently, we may think of the fibrant objects, the so-called *Kan complexes*, as "spaces" or as "homotopy types." Our aim is to tour some constructions on spaces that are semantic interpretations of well-known constructions in homotopy type theory [HoTT, Rij] following Hofmann-Streicher [HS2], Awodey–Warren [AW], Gambino–Garner [GG], and Voevodsky [KL]. Shulman has recently proved that homotopy

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I am so grateful to be one of innumerable members of our community who has had my life changed by Mike's generosity, mentorship, and friendship. It's an honor to have this chance to tell him how much he has and continues to mean to me.

type theory may be interpreted in any ∞-topos [Sh1],[1] so such constructions may also be described in myriad other settings [L, Re]. But for sake of concreteness we work in this classical model of classical homotopy theory.

1.1. **Path induction.** For any Kan complex A, there is a natural **path space factorization** of the diagonal map defined by exponentiation with the simplicial interval

$$\Delta^0 + \Delta^0 \rightarrowtail \Delta^1 \xrightarrow{\sim} \Delta^0 \qquad \rightsquigarrow \qquad A \overset{\text{refl}}{\underset{\sim}{\rightarrowtail}} A^{\Delta^1} \overset{(\mathrm{ev}_0,\mathrm{ev}_1)}{\twoheadrightarrow} A \times A \tag{1.1.1}$$

This data defines a reflexive binary relation on A, where "relation" is meant in the generalized sense, since the map $(\mathrm{ev}_0,\mathrm{ev}_1)\colon A^{\Delta^1} \twoheadrightarrow A \times A$ to the product is not necessarily a monomorphism. An element p in the fiber over a pair of points (x,y)—i.e., a *path* from x to y in A—witnesses the relation $p : x \sim y$. This motivates us to adopt the notation $x \sim y$ for the fiber and introduce the following alternate notation for the fibration $(\mathrm{ev}_0,\mathrm{ev}_1)\colon A^{\Delta^1} \twoheadrightarrow A \times A$, as an indexed family of its fibers:

$$\begin{tikzcd}
x \sim y \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\Delta^1} \arrow[d, two heads, "(x \sim y)_{x,y:A}"] \\
\Delta^0 \arrow[r, "{(x,y)}"'] & A \times A
\end{tikzcd}$$

As a consequence of Theorem 1.0.2, the left map of (1.1.1), the inclusion of constant paths, is a trivial cofibration in Quillen's model structure, and as such has the right lifting property with respect to an arbitrary Kan fibration ρ:

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, tail, "\text{refl}"', "\wr"] & E \arrow[d, two heads, "\rho = (E_b)_{b:B}"] \\
A^{\Delta^1} \arrow[r, "f"'] \arrow[ur, dashed] & B
\end{tikzcd}
\qquad \rightsquigarrow \qquad
\begin{tikzcd}
\operatorname{Map}(A^{\Delta^1}, E) \arrow[d, two heads, "\rho \circ - \circ \text{refl}", "\wr"'] \\
\operatorname{Map}(A,E) \times_{\operatorname{Map}(A,B)} \operatorname{Map}(A^{\Delta^1}, B) \arrow[u, dashed, bend left]
\end{tikzcd}$$

As a further consequence of Theorem 1.0.2, this lifting property can be *internalized*: the canonical map from the space $\operatorname{Map}(A^{\Delta^1}, E)$ of maps from the path space of A to E to the space $\operatorname{Map}(A,E) \times_{\operatorname{Map}(A,B)} \operatorname{Map}(A^{\Delta^1}, B)$ of commutative squares from refl to ρ is a trivial fibration. A section to this trivial fibration gives a continuous choice of solutions to lifting problems from refl to ρ. The fiber over a vertex (e,f) defines a contractible Kan complex; thus solutions to lifting problems from refl to ρ are "homotopically unique."

By pulling back along the codomain f, to solve lifting problems of the form above it suffices to consider right lifting problems against Kan fibrations over the path space A^{Δ^1} in which the codomain is the identity map:

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[rr, bend left, "e"] \arrow[d, tail, "\text{refl}"', "\wr"] & P \arrow[r] \arrow[d, two heads, "\chi"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "\rho"] \\
A^{\Delta^1} \arrow[r, equal] \arrow[ur, dashed, "J"] & A^{\Delta^1} \arrow[r, "f"'] & B
\end{tikzcd}
\qquad \rightsquigarrow \qquad
\begin{tikzcd}
\operatorname{Map}_{A^{\Delta^1}}(A^{\Delta^1}, P) \arrow[r] \arrow[d, two heads, "- \circ \text{refl}", "\wr"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Map}(A^{\Delta^1}, P) \arrow[d, two heads, "\chi \circ - \circ \text{refl}", "\wr"'] \\
\operatorname{Map}_A(A, P_{\text{refl}}) \arrow[r] \arrow[d] \arrow[u, dashed, bend left, "\text{path-ind}"] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Map}(A,P) \times_{\operatorname{Map}(A,A^{\Delta^1})} \operatorname{Map}(A^{\Delta^1}, A^{\Delta^1}) \arrow[d, two heads, "\pi"] \arrow[u, dashed, bend left] \\
\Delta^0 \arrow[r, "\text{id}"'] & \operatorname{Map}(A^{\Delta^1}, A^{\Delta^1})
\end{tikzcd}$$

[1] See [Rie] for an expository treatment, which cross-pollinated with these lecture notes.

Here the data of the lifting problem is given by a single map $d\colon A \to P$ that is a partial section of the fibration $\chi\colon P \twoheadrightarrow A^{\Delta^1}$ over the map refl.

$$\begin{tikzcd} & P \arrow[d, two heads, "(P_{x,y,p})_{x,y:A,p:x\sim y}"] \\ A \arrow[ur, dashed, "d"] \arrow[r, tail, "\sim", "\mathrm{refl}"'] & A^{\Delta^1} \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} P_{\mathrm{refl}} \arrow[r] \arrow[d, two heads, "(P_{a,a,\mathrm{refl}_a})_{a:A}"] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, two heads, "(P_{x,y,p})_{x,y:A,p:x\sim y}"] \\ A \arrow[u, dashed, bend left, "d"] \arrow[r, tail, "\sim", "\mathrm{refl}"'] & A^{\Delta^1} \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, dashed, "d"] \arrow[d, tail, "\mathrm{refl}"', "\wr"] & P \arrow[d, two heads, "(P_{x,y,p})_{x,y:A,p:x\sim y}"] \\ A^{\Delta^1} \arrow[r, equal] & A^{\Delta^1} \end{tikzcd}$$

As before there exists a continuous choice of solutions to this lifting problem given by a section to the trivial fibration displayed above-center which we call *path induction*, using an analogy first observed by Awodey and Warren [AW] and Gambino and Garner [GG]. The existence of this map proves the following proposition:

Proposition 1.1.2 (path induction, preliminary form)**.** *To define a section of a fibration over a path space A^{Δ^1}, it suffices to define a partial section over the subspace* refl: $A \xrightarrow{\sim} A^{\Delta^1}$ *of constant paths.*

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, tail, "\mathrm{refl}"', "\wr"] & P \arrow[d, two heads, "(P_{x,y,p})_{x,y:A,p:x\sim y}"] \\ A^{\Delta^1} \arrow[r, equal] \arrow[ur, dashed, "\text{path-ind}"] & A^{\Delta^1} \end{tikzcd}$$

Remark 1.1.3. A special case of fibrations over a path space are those that arise as pullbacks of fibrations over some other base. Later we'll refer to the process of pulling back a fibration along an arbitrary map as *substitution*. Path induction equally applies to define a partial section to E over f.

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, tail, "\mathrm{refl}"', "\wr"] & E \arrow[d, two heads, "(E_b)_{b:B}"] \\ A^{\Delta^1} \arrow[r, "f"'] \arrow[ur, dashed, "\text{path-ind}"] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r, "d"] \arrow[rr, bend left, "e"] \arrow[d, tail, "\mathrm{refl}"', "\wr"] & E_f \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "(E_b)_{b:B}"] \\ A^{\Delta^1} \arrow[r, equal] \arrow[ur, dashed, "\text{path-ind}"] & A^{\Delta^1} \arrow[r, "f"'] & B \end{tikzcd}$$

Path induction is a powerful proof technique, closely analogous to the principle of mathematical induction over the natural numbers. By analogy with that case, it also has constructive content, allowing the "recursive" definition of morphisms, though we follow convention and use the term "induction" for both induction and recursion.

Construction 1.1.4 (inversion)**.** Even though the simplicial interval Δ^1 has no symmetries, when A is a Kan complex, a path $p : x \sim y$ can be inverted to define a path $p^{-1}\colon y \sim x$. In other words, the path relation is symmetric as well as reflexive. This symmetry can be defined as a continuous function on path spaces by path-induction, which says it suffices to specify the inverse of a constant path, which we take to be a constant path:

$$\begin{tikzcd} A \arrow[r, tail, "\mathrm{refl}", "\sim"'] \arrow[d, tail, "\mathrm{refl}"', "\wr"] & A^{\Delta^1} \arrow[d, two heads, "(y\sim x)_{x,y:A}"] \\ A^{\Delta^1} \arrow[r, two heads, "(x\sim y)_{x,y:A}"'] \arrow[ur, dashed, "(-)^{-1}"'] & A \times A \end{tikzcd} \qquad (-)^{-1} := \text{path-ind}(\mathrm{refl} \mapsto \mathrm{refl})$$

The full principle of path induction is more general than the version stated in Proposition 1.1.2 along two axes. The first extension uses the fact that Quillen's model structure is *right proper*, meaning the weak equivalences are stable under pullback along fibrations. In particular, since the cofibrations are stable under pullback along arbitrary maps, the trivial cofibrations are stable under pullback along fibrations, something that is often called the *Frobenius condition* [GS]. The operation of pullback along a fibration can be thought of as introducing a trivial dependency and is referred to as *weakening*. A more general version of path-induction states that given a fibration over a weakening of a path space, to define a section it suffices to define a section over the subspace of constant paths.

Proposition 1.1.5 (path induction, intermediate form)**.** *Given a composable pair of fibrations $\chi\colon Q \twoheadrightarrow P$ and $\rho\colon P \twoheadrightarrow A^{\Delta^1}$ over a path space, to define a section of χ it suffices to define a partial section over the pullback along ρ of the subspace* $\mathrm{refl}\colon A \overset{\sim}{\rightarrowtail} A^{\Delta^1}$ *of constant paths.*

$$\begin{tikzcd}
A \arrow[d, tail, "\mathrm{refl}"', "\wr"] & P_{\mathrm{refl}} \arrow[l, two heads, "\rho_{\mathrm{refl}}"'] \arrow[r, "d"] \arrow[d, tail, "\mathrm{refl}_\rho"', "\wr"] \arrow[dl, phantom, very near start, "\llcorner"] & Q \arrow[d, two heads, "\chi=(Q_p)_{p:P}"] \\
A^{\Delta^1} & P \arrow[l, two heads, "\rho"] \arrow[r, equal] \arrow[ur, dashed, "\text{path-ind}"] & P
\end{tikzcd}$$

Construction 1.1.6 (concatenation)**.** When A is a Kan complex, paths $p : x \sim y$ and $q : y \sim z$ can be composed to define a path $p * q : x \sim z$. The composition function, establishing the transitivity of the path space relation, can be defined by path induction using the weakening of the reflexivity map displayed below-left:

$$\begin{tikzcd}
A \arrow[d, tail, "\mathrm{refl}"', "\wr"] & A^{\Delta^1} \arrow[l, two heads, "(\Sigma_{x:A} x\sim y)_{y:A}"'] \arrow[r, equal] \arrow[d, tail, "(1{,}\mathrm{refl})"', "\wr"] \arrow[dl, phantom, very near start, "\llcorner"] & A^{\Delta^1} \arrow[d, two heads, "(x\sim z)_{x,z:A}"] \\
A^{\Delta^1} & A^{\Lambda^2_1} \arrow[l, two heads, "(\Sigma_{x:A} x\sim y)_{y,z:A,q:y\sim z}"] \arrow[r, two heads, "(\Sigma_{y:A} x\sim y\sim z)_{x,z:A}"'] \arrow[ur, dashed, "*"'] & A \times A
\end{tikzcd} \qquad * := \text{path-ind}(p, \mathrm{refl} \mapsto p)$$

In summary, by path induction, the concatenation operation can be defined by specifying how to concatenate an arbitrary path $p : x \sim y$ by a constant path $\mathrm{refl}_y : y \sim y$, and we define concatenation with a constant path to be the identity function.

Note this construction of the concatenation function avoids the use of any higher simplices, though by homotopical uniqueness of solutions to lifting problems it is equivalent to the map defined by the more standard construction:

$$\begin{tikzcd}
A^{\Lambda^2_1} \arrow[r, tail, dashed, bend left, "\sim"] & A^{\Delta^2} \arrow[l, two heads] \arrow[r, two heads, "\circ"] & A^{\Delta^1}
\end{tikzcd}$$

Construction 1.1.7 (transport)**.** Fibrations $\rho\colon B \twoheadrightarrow A$ are accompanied by path lifting or "transport" operations that lift a path $p : x \sim y$ in the base space to a continuous map $\mathrm{tr}_p\colon B_x \to B_y$ between the fibers.

As before, this operation can be expressed as a continuous function between spaces, where the domain is formed by pulling the domain projection $e_0\colon A^{\Delta^1} \twoheadrightarrow A$ back along ρ:

$$\begin{tikzcd}
A \arrow[d, tail, "\mathrm{refl}"', "\wr"] & B \arrow[l, two heads, "(B_a)_{a:A}"'] \arrow[r, equal] \arrow[d, tail, "\mathrm{refl}\times 1"', "\wr"] \arrow[dl, phantom, very near start, "\llcorner"] & B \arrow[d, two heads, "\rho=(B_y)_{y:A}"] \\
A^{\Delta^1} & A^{\Delta^1} \times_A B \arrow[l, two heads, "(B_x)_{x,y:A,p:x\sim y}"] \arrow[r, two heads, "(\Sigma_{x:A}(x\sim y)\times B_x)_{y:A}"'] \arrow[ur, dashed, "\mathrm{tr}"'] & A
\end{tikzcd} \qquad \mathrm{tr} := \text{path-ind}(\mathrm{refl}, u \mapsto u)$$

In summary, the transport operation is defined by path induction by declaring that transport along a constant path is the identity function.

1.2. **Contexts.** The homotopy type theoretic principle of path induction is yet stronger than this, because A might be a "space in an arbitrary context Γ." We can perform an analogous construction starting from an arbitrary Kan fibration $\rho\colon A \twoheadrightarrow \Gamma$, whether or not the base Γ is a Kan complex. First form the path space factorizations for both A and Γ

(1.2.1)
$$\begin{tikzcd}
A \arrow[rrr] \arrow[dddd, two heads] \arrow[dr, tail, "\sim"] & & & A^{\Delta^1} \arrow[r] \arrow[dddd, two heads] & A \times A \arrow[dddd, two heads] \\
& \Delta^1 \pitchfork_\Gamma A \arrow[urr, dotted] \arrow[dr, two heads] \arrow[dddl, dotted, two heads] & & & \\
& & A \times_\Gamma A \arrow[uurr, dotted, tail] \arrow[ddll, dotted, two heads] & & \\
& & & & \\
\Gamma \arrow[rrr] & & & \Gamma^{\Delta^1} \arrow[r] & \Gamma \times \Gamma
\end{tikzcd}$$

and then pull back the top factorization so that it lies in the slice over Γ. This constructs a factorization of the fibered diagonal, in the slice over Γ, using the cotensor with the simplicial interval in the slice over

Γ. If Γ and A are not themselves fibrant, their path space factorizations lose the homotopical properties deployed above. However, these properties are restored in the fibered path space factorization.

Proposition 1.2.2. *For any fibration $\rho\colon A \twoheadrightarrow \Gamma$, the natural maps*

$$A \overset{\mathrm{refl}}{\underset{\sim}{\rightarrowtail}} \Delta^1 \pitchfork_\Gamma A \xrightarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} A \times_\Gamma A$$

over Γ give a factorization of the fibered diagonal map $(1,1)\colon A \to A \times_\Gamma A$ in $\mathsf{s}\mathcal{S}\mathsf{et}_{/\Gamma}$ as a trivial cofibration followed by a fibration. Moreover, this construction is weakly stable under substitution, meaning the pullback of this factorization along any $f : \Delta \to \Gamma$ is isomorphic to the analogous factorization in $\mathsf{s}\mathcal{S}\mathsf{et}_{/\Delta}$.

Proof. The endpoint evaluation map is a pullback of the Leibniz exponential of the inclusion $\partial\Delta^1 \rightarrowtail \Delta^1$ with the fibration $\rho\colon A \twoheadrightarrow \Gamma$. Since Quillen's model structure is simplicial, this latter map is a fibration, and so the former one is as well:

$$\begin{tikzcd}
A \arrow[rrr] \arrow[dr, tail, "\sim"] \arrow[ddd, two heads] & & & A^{\Delta^1} \arrow[rr] \arrow[dr, two heads] \arrow[ddd, two heads] & & A \times A \arrow[ddd, two heads] \\
& \Delta^1 \pitchfork_\Gamma A \arrow[urr, dotted] \arrow[dr, two heads] \arrow[ddl, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & & \bullet \arrow[ur, dotted] \arrow[ddl, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \\
& & A \times_\Gamma A \arrow[urr, dotted] \arrow[dll, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & & \\
\Gamma \arrow[rrr] & & & \Gamma^{\Delta^1} \arrow[rr] & & \Gamma \times \Gamma
\end{tikzcd}$$

Similarly, each individual endpoint projection $e_0, e_1 : \Delta^1 \pitchfork_\Gamma A \to A$ is a pullback of the Leibniz exponential of the corresponding endpoint inclusion $0,1 : \Delta^0 \overset{\sim}{\rightarrowtail} \Delta^1$ with $\rho\colon A \twoheadrightarrow \Gamma$. Since this former map is a trivial fibration, the endpoint inclusions are trivial fibrations, and it follows that their common section $\mathrm{refl}\colon A \overset{\sim}{\rightarrowtail} \Delta^1 \pitchfork_\Gamma A$ is a trivial cofibration.

Moreover, since pullback is a simplicially enriched right adjoint, this construction is stable under pullback along any $f : \Delta \to \Gamma$ between Kan complexes.

$$\begin{tikzcd}
A_f \arrow[rrr, dotted] \arrow[dr, tail, "\sim"] \arrow[dddr, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & & A \arrow[dr, tail, "\sim"] \arrow[dddr, two heads] & & \\
& \Delta^1 \pitchfork_\Delta A_f \arrow[rrr, dotted] \arrow[dr, two heads] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & & \Delta^1 \pitchfork_\Gamma A \arrow[dr, two heads] \arrow[dd, two heads] & \\
& & A_f \times_\Delta A_f \arrow[rrr, dotted] \arrow[dl, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & & A \times_\Gamma A \arrow[dl, two heads] \\
& \Delta \arrow[rrr, dotted, "f"'] & & & \Gamma &
\end{tikzcd}$$

□

Once more the inclusion of constant paths is a trivial fibration so we have a lifting property exactly as above. This proves the general form of the path induction principle.

Proposition 1.2.3 (path induction, final form)**.** *Given a composable pair of fibrations $\chi\colon Q \twoheadrightarrow P$ and $\rho\colon P \twoheadrightarrow \Delta^1 \pitchfork_\Gamma A$ over a path space in any context Γ, to define a section of χ it suffices to define a partial section over the pullback along ρ of the subspace* $\mathrm{refl}\colon A \overset{\sim}{\rightarrowtail} \Delta^1 \pitchfork_\Gamma A$ *of constant paths.*

$$\begin{tikzcd}
P_{\mathrm{refl}} \arrow[r, two heads, "\rho_{\mathrm{refl}}"] \arrow[d, tail, "\mathrm{refl}_\rho"', "\wr"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "\mathrm{refl}", "\wr"'] \\
P \arrow[r, two heads, "\rho"'] & \Delta^1 \pitchfork_\Gamma A
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
P_{\mathrm{refl}} \arrow[r, "d"] \arrow[d, tail, "\mathrm{refl}_\rho"', "\wr"] & Q \arrow[d, two heads, "\chi=(Q_p)_{p:P}"] \\
P \arrow[r, equal] \arrow[ur, dashed, "\text{path-ind}"] & P
\end{tikzcd}$$

Remark 1.2.4. Note via Proposition 1.2.2, Proposition 1.2.3 is a precise analogue of Proposition 1.1.5 just interpreted in the slice category $\mathsf{s}\mathcal{S}\mathsf{et}_{/\Gamma}$ instead of in $\mathsf{s}\mathcal{S}\mathsf{et}$. Crucially, all of the properties of Theorem 1.0.2 are stable under slicing, meaning inherited by all sliced categories $\mathsf{s}\mathcal{S}\mathsf{et}_{/\Gamma}$. In particular, this is why we described Quillen's model structure as *simplicial* (enriched over Quillen's model structure on simplicial sets) rather than *cartesian closed* (enriched over itself), because the former property is stable under slicing while the latter is not.

1.3. **Propositions as homotopy types.** The category of simplicial sets is *locally cartesian closed*, meaning that for any $f : \Delta \to \Gamma$ the composition functor has two right adjoints

$$\begin{tikzcd} s\mathcal{S}\mathrm{et}_{/\Delta} \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & s\mathcal{S}\mathrm{et}_{/\Gamma} \arrow[l, "f^*"'] \end{tikzcd} \tag{1.3.1}$$

defined by pullback along f and pushforward along f, respectively. Since $s\mathcal{S}\mathrm{et}$ has a terminal object Δ^0, this implies that $s\mathcal{S}\mathrm{et} \cong s\mathcal{S}\mathrm{et}_{/\Delta^0}$ and indeed any slice $s\mathcal{S}\mathrm{et}_{/\Gamma}$ is cartesian closed and has all finite limits.

Lemma 1.3.2. *When $f : \Delta \twoheadrightarrow \Gamma$ is a fibration, all three adjoints $\Sigma_f \dashv f^* \dashv \Pi_f$ preserve fibrations.*

They fibration hypothesis is not needed for the pullback functor, but is needed for the other two adjoints.

Proof. The first two statements are the familiar closure of fibrations under composition and pullback. The final statement is a consequence of the Frobenius property: since trivial cofibrations are stable under pullback along fibrations, by transposing, fibrations are stable under pushforward along fibrations. □

Notation 1.3.3. The following notation is intended to make the composition, pullback, and pushforward operations more legible.

- Fibrations will be depicted as an indexed family of spaces

$$\begin{tikzcd} B \arrow[d, two heads, "(B_a)_{a:A}"] \\ A \end{tikzcd} \qquad \begin{tikzcd} B_a \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, two heads, "(B_a)_{a:A}"] \\ \Gamma \arrow[r, "a"'] & A \end{tikzcd}$$

  particularly when the fibration is regarded as a fibrant object in the slice category $s\mathcal{S}\mathrm{et}_{/A}$. Here B_a is also notation for the fiber over a generalized element $a \colon \Gamma \to A$ abbreviated $a : A$. We refer to a fibration of this form as a "space B in context A."
- With this notation, pullbacks are denoted as follows:

$$\begin{tikzcd} B_f \arrow[r] \arrow[d, two heads, "(B_{fc})_{c:C}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, two heads, "(B_a)_{a:A}"] \\ C \arrow[r, "f"'] & A \end{tikzcd}$$

  In the identification of the fibers, the element fc is substituted for the element a, which is why pullbacks are also called *substitutions*.
- When A is a Kan complex, the functor Σ_A defined by composing with $! \colon A \twoheadrightarrow \Delta^0$ sends a fibration $(B_a)_{a:A}$ to a space $\Sigma_{a:A} B_a$, namely the domain B of the fibration.

$$\begin{tikzcd} \Sigma_{a:A} B_a \arrow[d, two heads, "(B_a)_{a:A}"'] & & \Sigma_{a:A} B_a \arrow[d, two heads, "!"] \\ A \arrow[rr, two heads, "!"'] & \mapsto & \Delta^0 \end{tikzcd}$$

  More generally, when A is a space $(A_\gamma)_{\gamma:\Gamma}$ in context Γ, the composition functor sends a fibration $(B_a)_{a:A}$ to the fibration

$$\begin{tikzcd} \Sigma_{a:A} B_a \arrow[d, two heads, "(B_a)_{a:A}"'] & & \Sigma_{\gamma:\Gamma}\Sigma_{a:A_\gamma} B_a \arrow[d, two heads, "(\Sigma_{a:A_\gamma} B_a)_{\gamma:\Gamma}"] \\ A \arrow[rr, two heads, "(A_\gamma)_{\gamma:\Gamma}"'] & \mapsto & \Gamma \end{tikzcd}$$

Since the effect of composing a pair of fibrations is to "sum up the fibers" we also refer to the composition functor as *summation*.

- When A is a Kan complex, the functor Π_A defined by pushforward along $!\colon A \twoheadrightarrow \Delta^0$ sends a fibration $(B_a)_{a:A}$ to a space $\Pi_{a:A} B_a$, namely the space $\mathrm{Map}_A(A, B)$ of sections of the fibration.

$$\begin{tikzcd} \Sigma_{a:A} B_a \arrow[d, two heads, "(B_a)_{a:A}"'] & \mapsto & \Pi_{a:A} B_a \arrow[d, two heads, "!"] \\ A \arrow[rr, two heads, "!"'] & & \Delta^0 \end{tikzcd}$$

More generally, when A is a space $(A_\gamma)_{\gamma:\Gamma}$ in context Γ, the pushforward functor sends a fibration $(B_a)_{a:A}$ to the fibration

$$\begin{tikzcd} \Sigma_{a:A} B_a \arrow[d, two heads, "(B_a)_{a:A}"'] & \mapsto & \Sigma_{\gamma:\Gamma}\Pi_{a:A_\gamma} B_a \arrow[d, two heads, "(\Pi_{a:A_\gamma} B_a)_{\gamma:\Gamma}"] \\ A \arrow[rr, two heads, "(A_\gamma)_{\gamma:\Gamma}"'] & & \Gamma \end{tikzcd}$$

The notations just introduced are "stable under substitution," commuting with pullbacks. For instance, the fibers over γ of the composition or pushforward along a fibration $(A_\gamma)_{\gamma:\Gamma}$ agrees with the spaces obtained by composition or pushforward over the space A_γ, which we state in the case of a global element but the proof applies equally to a generalized element $\gamma\colon \Delta \to \Gamma$.

Lemma 1.3.4. *For any element $\gamma\colon \Delta^0 \to \Gamma$,*

$$\begin{tikzcd} \Sigma_{a:A_\gamma} B_a \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma_{\gamma:\Gamma}\Sigma_{a:A_\gamma} B_a \arrow[d, two heads, "(\Sigma_{a:A_\gamma} B_a)_{\gamma:\Gamma}"] \\ \Delta^0 \arrow[r, "\gamma"'] & \Gamma \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Pi_{a:A_\gamma} B_a \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma_{\gamma:\Gamma}\Pi_{a:A_\gamma} B_a \arrow[d, two heads, "(\Pi_{a:A_\gamma} B_a)_{\gamma:\Gamma}"] \\ \Delta^0 \arrow[r, "\gamma"'] & \Gamma \end{tikzcd}$$

Proof. The space A_γ is defined by the pullback square

$$\begin{tikzcd} A_\gamma \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "(A_\gamma)_{\gamma:\Gamma}"] \\ \Delta^0 \arrow[r, "\gamma"'] & \Gamma \end{tikzcd}$$

By the composition and cancellation lemma for pullback squares, for any fibration over A, the result of composing along the inclusion $(A_\gamma)_{\gamma:\Gamma}$ and then pulling back along γ is isomorphic to the result of pulling back along $A_\gamma \to A$ and then summing over A_γ.[2] The corresponding operations in the case of the pushfoward are right adjoints to these (though starting from the other corner of the square and traversing it in the other direction) and thus must also be isomorphic. This constructs the so-called *Beck-Chevalley isomorphisms*. □

The adjoint triples $\Sigma_f \dashv f^* \dashv \Pi_f$ can be used to construct spaces in the empty context or fibrations in a general context Γ whose global elements — vertices in the former case and sections in the latter case — provide data witnessing the proof of some proposition. Thus, these spaces exhibit mathematical propositions as homotopy types. Since the inputs to our constructions are features that are equally present in slices $s\mathcal{S}\mathrm{et}_{/\Gamma}$ we may leave the context Γ implicit in our notation, representing a fibration $\rho\colon A \twoheadrightarrow \Gamma$ as a space "A".

Construction 1.3.5 (contractibility)**.** Fix a Kan complex A. Under the notations just introduced, the path space fibration $(\mathrm{ev}_0, \mathrm{ev}_1)\colon A^{\Delta^1} \twoheadrightarrow A \times A$ is also denoted by $(x \sim y)_{x,y:A}$. The pushforward along the projection

[2] Importantly for the second part of this proof, if we don't care whether the resulting map is a fibration, it doesn't matter whether any maps in the pullback square are fibrations.

away from the second coordinate defines the fibration $(\Pi_{y:A} x \sim y)_{x:A}$, and then the sum along the projection away from the first coordinate defines the space $\Sigma_{x:A}\Pi_{y:A} x \sim y$.

$$\begin{tikzcd}
A^{\Delta^1} \arrow[d, two heads, "(x\sim y)_{x,y:A}"] & \mapsto & \Sigma_{x:A}\Pi_{y:A} x \sim y \arrow[d, two heads, "(\Pi_{y:A} x\sim y)_{x:A}"] & \mapsto & \Sigma_{x:A}\Pi_{y:A} x \sim y \arrow[d, two heads, "!"] \\
A \times A & & A & & \Delta^0
\end{tikzcd}$$

A global element of $\Sigma_{x:A}\Pi_{y:A} x \sim y$ defines firstly an element $a : A$ together with an element of the fiber $\Pi_{y:A} a \sim y$, which is a section of the based path space fibration. Together this data proves the contractibility of A, and thus we define

$$\mathsf{isContr}(A) := \Sigma_{x:A}\Pi_{y:A} x \sim y$$

An example of a contractible space is a based path space. Fix a point $a : A$ and consider the space $\Sigma_{y:A} a \sim y$ defined by

(1.3.6)
$$\begin{tikzcd}
\Sigma_{y:A} a \sim y \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\Delta^1} \arrow[d, two heads, "(x\sim y)_{x,y:A}"] \\
\Delta^0 \arrow[r, "a"'] & A
\end{tikzcd}$$

A proof that $\Sigma_{y:A} a \sim y$ is contractible defines a global element of $\mathsf{isContr}(\Sigma_{y:A} a \sim y)$, but we can do better. When we "fix a point $a : A$" or "let a be a point A," we are introducing the space A as the context. Thus, our based path spaces are more properly thought of as fibrations $(\Sigma_{y:A} a \sim y)_{a:A} \in s\mathcal{S}\mathsf{et}_{/A}$.

By Remark 1.2.4, the construction of Construction 1.3.5 can be interpreted in any context. When applied to a fibration $(A_\gamma)_{\gamma:\Gamma}$ it defines a fibration $\mathsf{isContr}((A_\gamma)_{\gamma:\Gamma}) \cong (\mathsf{isContr}(A_\gamma))_{\gamma:\Gamma} \in s\mathcal{S}\mathsf{et}_\Gamma$ whose fiber over $\gamma : \Gamma$ is the space $\mathsf{isContr}(A_\gamma)$ by Lemma 1.3.4. When applied to the fibration $(\Sigma_{y:A} a \sim y)_{a:A}$, we obtain a fibration $(\mathsf{isContr}(\Sigma_{x:A} a \sim x))_{a:A} \in s\mathcal{S}\mathsf{et}_{/A}$ admitting a section, which proves the contractibility of the based path spaces, simultaneously and continuously for all $a : A$. Indeed:

Lemma 1.3.7. *A fibration $\rho\colon A \twoheadrightarrow \Gamma$ is a trivial fibration if and only if $\mathsf{isContr}(\rho) \in s\mathcal{S}\mathsf{et}_{/\Gamma}$ has a section.*

Proof. Here $\mathsf{isContr}(\rho)\colon \mathsf{isContr}_\Gamma(A) \twoheadrightarrow \Gamma$ is the result of interpreting Construction 1.3.5 applied to $(A_\gamma)_{\gamma:\Gamma}$ in the slice over Γ: i.e., this is the fibration $(\mathsf{isContr}(A_\gamma))_{\gamma:\Gamma}$. From the definition

$$\mathsf{isContr}(\rho) := \Sigma_\rho \Pi_{\pi_1} \Delta^1 \pitchfork_\Gamma A \ \in \ s\mathcal{S}\mathsf{et}_{/\Gamma},$$

we see that a section

$$\begin{tikzcd}
\mathsf{isContr}_\Gamma(A) \arrow[d, two heads, "\Pi_{\pi_1}\Delta^1 \pitchfork_\Gamma A"] \\
A \arrow[u, dashed, bend left, "\gamma"] \arrow[d, two heads, "\rho"] \\
\Gamma \arrow[u, dashed, bend left, "s"]
\end{tikzcd}$$

provides the data of a section s to ρ together with a homotopy $\gamma\colon s\rho \sim \mathrm{id}_A$ over Γ.

$$\begin{tikzcd}
& \Delta^1 \pitchfork_\Gamma A \arrow[d, two heads] \\
A \arrow[ur, dashed, bend left, "\gamma"] \arrow[r, "(s\rho, 1_A)"] \arrow[d, two heads, "\rho"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \times_\Gamma A \arrow[d, "\pi_1"] \\
\Gamma \arrow[r, "s"'] & A
\end{tikzcd}$$

Thus, ρ is a weak equivalence and hence a trivial fibration, and conversely, trivial fibrations admit such data. $\square$

Warning 1.3.8. In the category of simplicial sets, a fibration between fibrant objects is a trivial fibration if and only if its fibers are contractible, but in a general model category, the fiberwise contractibility condition is weaker than contractibility in the slice. As Christensen explains, it follows from this fact that in simplicial sets, various properties can be checked "in the empty context," meaning in the slice over Δ^0 [Ch, §5.2].

Construction 1.3.9 (equivalence)**.** Consider a map $f\colon A \to B$ between Kan complexes. The familiar mapping path space construction defines a fibration:

(1.3.10)
$$\begin{tikzcd} \Sigma_{a:A}\Sigma_{b:B} f a \sim b \arrow[r] \arrow[d, two heads, "(fa\sim b)_{a:A,b:B}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\Delta^1} \arrow[d, two heads, "(x\sim y)_{x,y:B}"] & & \Sigma_{b:B}\Sigma_{a:A} f a \sim b \arrow[d, two heads, "(\Sigma_{a:A} fa\sim b)_{b:B}"] \\ A \times B \arrow[r, "f\times \mathrm{id}"'] & B \times B & \rightsquigarrow & B \end{tikzcd}$$

whose fibers are the spaces

$$\mathsf{fib}_f\, b := \Sigma_{a:A} f a \sim b.$$

These allow us to define the space

$$\mathsf{isEquiv}(f) := \Pi_{b:B}\mathsf{isContr}(\mathsf{fib}_f\, b).$$

As the terminology suggests, a point in the space $\mathsf{isEquiv}(f)$ proves that $f\colon A \to B$ is a homotopy equivalence between Kan complexes. Such a term provides the data of a section to the fibration below-left:

$$\begin{tikzcd} \Sigma_{b:B}\mathsf{isContr}(\mathsf{fib}_f\, b) \arrow[d, two heads, "(\mathsf{isContr}(\mathsf{fib}_f\, b))_{b:B}"] & & \Sigma_{b:B}\mathsf{fib}_f\, b \arrow[d, two heads, "(\mathsf{fib}_f\, b)_{b:B}"] \\ B \arrow[u, dashed, bend left] & \rightsquigarrow & B \arrow[u, dashed, bend left] \end{tikzcd}$$

Passing to the center of contraction, this defines a section to the fibration above-right, which gives the data of a continuous function $g\colon B \to A$ together with a homotopy $\beta : \Pi_{b:B} f g b \sim b$. The remaining data—which witnesses that the fibers are contractible, not just inhabited—can be used to construct a second homotopy $\alpha : \Pi_{a:A} g f a \sim a$.

Again, this construction can be interpreted in any context:

Lemma 1.3.11. *A map $f\colon A \to B$ between fibrations over Γ is a weak equivalence if and only if* $\mathsf{isEquiv}(f) \in s\mathcal{S}\mathsf{et}_{/\Gamma}$ *has a section.*

Proof. A map is a weak equivalence if and only if its replacement by a weakly equivalent fibrant is a trivial fibration. In the case of a map $f\colon A \to B$ over Γ, this construction can be implemented in the slice over Γ, by interpreting the construction (1.3.10) in the slice over Γ (i.e., using the fibered path space and fibered products in place of the ordinary map space and ordinary products).

$$\begin{tikzcd} \Sigma_{a:A}\Sigma_{b:B} f a \sim_\Gamma b \arrow[r] \arrow[d, two heads, "(fa\sim_\Gamma b)_{a:A,b:B}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^1 \pitchfork_\Gamma B \arrow[d, two heads, "(x\sim_\Gamma y)_{x,y:B}"] & & \Sigma_{b:B}\Sigma_{a:A} f a \sim_\Gamma b \arrow[d, two heads, "(\mathsf{fib}_f\, b)_{b:B} := (\Sigma_{a:A} fa\sim_\Gamma b)_{b:B}"] \\ A \times_\Gamma B \arrow[r, "f\times \mathrm{id}"'] & B \times_\Gamma B & \rightsquigarrow & B \end{tikzcd}$$

By Lemma 1.3.7, the fibration $(\mathsf{fib}_f\, b)_{b:B}$ is a trivial fibration if and only if $\mathsf{isContr}(\mathsf{fib}_f\, b)_{b:B} \in s\mathcal{S}\mathsf{et}_{/B}$ has a section, or equivalently if and only if the pushforward $(\Pi_{b:B_\gamma}\mathsf{isContr}(\mathsf{fib}_f\, b))_{\gamma:\Gamma} \in s\mathcal{S}\mathsf{et}_{/\Gamma}$, has a section. This is exactly how we defined the fibration $\mathsf{isEquiv}(f) \in s\mathcal{S}\mathsf{et}_{/\Gamma}$. □

The construction of the space $\mathsf{isEquiv}(f)$ has an interesting feature, whose proof is too involved to present here:

Lemma 1.3.12. *For any $f\colon A \to B$, if the space* $\mathsf{isEquiv}(f)$ *is inhabited then it is contractible.*

Construction 1.3.13 (equivalences)**.** For spaces A and B, define a space

$$A \simeq B := \Sigma_{f:A\to B}\mathsf{isEquiv}(f).$$

Here we're using $A \to B$ as alternate notation for the mapping space B^A. The space $A \simeq B$ has a natural projection map to the mapping space $A \to B$, which is a homotopy monomorphism by Lemma 1.3.12.

2. Univalent universes of spaces

The previous constructions were defined in the context of arbitrary spaces A and B. We can regard these as constructions in the slice where we use a *universe* $\mathcal{U}$ of small spaces to define our context. Our next task is to define this space $\mathcal{U}$.

2.1. Universes. As a topos, the category $s\mathcal{S}et$ has a *subobject classifier*, a monomorphism $\top\colon \Delta^0 \to \Omega$ that is universal in the sense that it represents the functor $\mathrm{Sub}\colon s\mathcal{S}et^{\mathrm{op}} \to \mathcal{S}et$ that sends a simplicial set to its set of subobjects up to isomorphism, and acts on morphisms by pullback. Unfolding this characterization, the defining universal property tells us that morphisms $A\colon \Gamma \to \Omega$ correspond to subobjects of Γ up to isomorphism, formed by taking the pullback:[3]

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^0 \arrow[d, "\top"] \\ \Gamma \arrow[r, "A"'] & \Omega \end{tikzcd}$$

By the Yoneda lemma, this universal property tells us how to construct the simplicial set Ω: its n-simplices are the simplicial subsets of Δ^n. The subobject classifier Ω contains the nerve of the free-living isomorphism as a simplicial subspace whose simplices encode all subobjects of the form $\Delta^k \rightarrowtail \Delta^n$. But it has many additional simplices corresponding to subobjects formed as unions of standard simplices.

We might try to define a classifier for small[4] fibrations—another isomorphism-invariant pullback-stable family of maps—similarly as the representing object for an analogously-defined functor, but this analogous functor fails to be representable, since it does not carry colimits to limits. The culprit is the potential existence of non-trivial automorphisms between non-monomorphisms with fixed base. So instead our universes will "classify" the family of maps in a more delicate sense[5] that we won't fully elaborate on here [Sh1], which will at least imply that every small fibration arises (non-uniquely) as a pullback of the universal fibration

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d, two heads, "v"] \\ \Gamma \arrow[r, "A"'] & \mathcal{U} \end{tikzcd}$$

By the Yoneda lemma, we might try to define $\mathcal{U}$ to be the simplicial set whose n-simplices are small fibrations over Δ^n, but since the categorical pullback is not strictly functorial, this defines a groupoid valued pseudofunctor on $\mathbb{\Delta}^{\mathrm{op}}$ rather than a strict simplicial set. Instead, we'll define $\mathcal{U}$ as a simplicial subset of a simplicial set $\mathcal{V}$ that classifies maps with small fibers, which may be constructed by a general method, originally due to Hofmann and Streicher [HS1] introduced here using a particularly slick construction due to Steve Awodey [A2].

Let κ be an infinite regular cardinal,[6] and write $\mathrm{set} \in \mathcal{C}at$ for a full subcategory of sets containing at least one set of each cardinality $\lambda < \kappa$ and at most κ many sets of each cardinality.

Definition 2.1.1. Consider the covariant functor $\mathbb{\Delta}_{/\bullet}\colon \mathbb{\Delta} \to \mathcal{C}at$ sending $[n] \in \mathbb{\Delta}$ to the slice category $\mathbb{\Delta}_{/[n]}$ and the simplicial operators $\alpha\colon [n] \to [m]$ to the composition functors $\Sigma_\alpha\colon \mathbb{\Delta}_{/[n]} \to \mathbb{\Delta}_{/[m]}$. These slice categories are the *categories of elements* of the simplicial sets Δ^n and the left Kan extension of this functor along the

[3] Note the construction of taking the pullback of a cospan of this form is only well-defined up to isomorphism over Γ. In any case, if we failed to identity isomorphic subobjects, this construction of the functor Sub would (i) be valued in large sets and (ii) would fail to be strictly functorial.

[4] Here "small" means that the cardinality of the fibers is bounded by some infinite regular cardinal κ, which is an implicit parameter in everything that follows.

[5] The key property is that the classifying squares can be extended along monomorphisms between the bases, a property known as "realignment" [GSS].

[6] As the title of their article suggests, Hofmann and Streicher prefer to work with an inaccessible cardinal [HS1], but for the purposes of obtaining a κ-small map classifier it suffices to use a regular cardinal κ larger than the cardinality of the morphisms in the indexing category.

Yoneda embedding defines the category of elements functor and its right adjoint:

$$\begin{tikzcd} & \mathbb{\Delta} \arrow[dl, hook', "\yo"'] \arrow[dr, "\mathbb{\Delta}_{/\bullet}"] & \\ s\mathcal{S}\mathrm{et} \arrow[rr, bend left, "\int:=\mathrm{colim}_{\mathbb{\Delta}_{/\bullet}}"] \arrow[rr, phantom, "\perp"] & & \mathcal{C}\mathrm{at} \arrow[ll, bend left, "N_{\int}:=\hom(\mathbb{\Delta}_{/\bullet},-)"] \end{tikzcd}$$

The κ-small map classifier is then defined by applying the right adjoint to the opposite of the forgetful functor from pointed sets to sets:

$$\begin{tikzcd} \tilde{\mathcal{V}} \arrow[d, "\varpi"'] & := & N_{\int}\mathsf{set}_*^{\mathrm{op}} \arrow[d] \\ \mathcal{V} & := & N_{\int}\mathsf{set}_* \end{tikzcd}$$

Explicitly, an n-simplex in $\mathcal{V}$ is a presheaf $A\colon \mathbb{\Delta}_{/[n]}^{\mathrm{op}} \to \mathsf{set}$. As noted above, it's insufficient to define $\mathcal{V}_n$ to be the set of κ-small maps over Δ^n because the action of a simplicial operators $\alpha\colon \Delta^m \to \Delta^n$, by pullback, is only pseudofunctorial. However, under the equivalence of categories[7]

$$\begin{tikzcd} \mathsf{sset}_{/\Delta^n} \arrow[d, "\alpha^*"'] & \simeq & \mathsf{set}^{\mathbb{\Delta}_{/[n]}^{\mathrm{op}}} \arrow[d, "-\cdot\alpha_!"] \\ \mathsf{sset}_{/\Delta^m} & \simeq & \mathsf{set}^{\mathbb{\Delta}_{/[m]}^{\mathrm{op}}} \end{tikzcd}$$

the pullback action is replaced by precomposition with the composition functor $\alpha_!\colon \mathbb{\Delta}_{/[m]} \to \mathbb{\Delta}_{/[n]}$, which is strictly functorial, defining a large simplicial set $\mathcal{V} \in s\mathcal{S}\mathrm{et}$. An element $A\colon \Delta^n \to \mathcal{V}$ defines a small map over Δ^n whose fiber over $\alpha \in (\Delta^n)_m$ is the set A_α defined by the functor $A\colon \mathbb{\Delta}_{/[n]}^{\mathrm{op}} \to \mathsf{set}$.

An n-simplex in $\tilde{\mathcal{V}}$ is a presheaf $A\colon \mathbb{\Delta}_{/[n]}^{\mathrm{op}} \to \mathsf{set}_*$ valued in pointed sets. Since the indexing category $\mathbb{\Delta}_{/[n]}$ has a terminal object $\mathrm{id}_{[n]}$, an n-simplex in $\tilde{\mathcal{V}}$ is equally given by the data of $A\colon \Delta^n \to \mathcal{V}$ together with a section

$$\begin{tikzcd} & \tilde{\mathcal{V}} \arrow[d, "\varpi"] \\ \Delta^n \arrow[r, "A"'] \arrow[ur, dashed, "a"] & \mathcal{V} \end{tikzcd}$$

2.2. **A universal Kan fibration.** The technique of Hofmann-Streicher universes can be used to define a universal Kan fibration, which is the key ingredient in the simplicial model of univalent foundations. This is not the approach taken to defining the universe in [KL] but was quickly noted as alternative possible route; see [Ci] or [Str].

The Hofmann-Streicher universe may be restricted to define the universal Kan fibration by taking $\mathcal{U} \subset \mathcal{V}$ to be the simplicial subset spanned by those small maps over simplices that are Kan fibrations. One then defines $\upsilon\colon \tilde{\mathcal{U}} \to \mathcal{U}$ to be the pullback

$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[r] \arrow[d, "\upsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{V}} \arrow[d, "\varpi"] \\ \mathcal{U} \arrow[r, hook] & \mathcal{V} \end{tikzcd} \tag{2.2.1}$$

[7]More generally, the slice category over any presheaf X is equivalent to the category of presheaves on the category of elements of X. In particular, "$\mathbb{\Delta}_{/[n]}^{\mathrm{op}}$" should be read as an abbreviation for $(\mathbb{\Delta}_{/[n]})^{\mathrm{op}}$, the slice category $\mathbb{\Delta}_{/[n]}$ being the category of elements of the simplicial set Δ^n.

Crucially, the Kan fibrations of simplicial sets are *local*: a map $p\colon E \twoheadrightarrow B$ is a Kan fibration if and only if for all n and all $b\colon \Delta^n \to B$, the pullback defines a Kan fibration. This follows from the fact that the fibrations are characterized by a right lifting property against maps with representable codomains:

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, tail, "\wr"'] & \bullet \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
\Delta^n \arrow[r, equal] \arrow[ur, dashed] & \Delta^n \arrow[r, "b"'] & B
\end{tikzcd}$$

Consequently:

Corollary 2.2.2. *$\pi\colon \tilde{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is a Kan fibration.*

Proof. By construction, each pullback

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d, two heads, "\upsilon"] \\
\Delta^n \arrow[r, "A"'] & \mathcal{U}
\end{tikzcd}$$

is a Kan fibration. □

Another reflection of this locality is the following:

Lemma 2.2.3. *Let $\rho\colon A \to \Gamma$ be a small map and consider any classifying square*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\rho"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{V}} \arrow[d, "\varpi"] \\
\Gamma \arrow[r, "A"'] & \mathcal{V}
\end{tikzcd}$$

Then ρ is a Kan fibration if and only if the classifying square factors through (2.2.1).

Proof. If ρ is a pullback of $\upsilon\colon \tilde{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ it is clearly a Kan fibration. Conversely, if p is a Kan fibration then so is its restriction along any $\gamma\colon \Delta^n \to \Gamma$. Recall $\mathcal{U}$ is defined as a simplicial subset of $\mathcal{V}$, so $A\colon \Gamma \to \mathcal{V}$ factors through $\mathcal{U} \hookrightarrow \mathcal{V}$ just when for each $\gamma\colon \Delta^n \to \Gamma$ the corresponding map $A_\gamma\colon \Delta^n \to \mathcal{V}$ so factors. But since a Kan fibration pulls back to Kan fibrations, this means the corresponding elements are in the simplicial subset. □

Corollary 2.2.4. *Any Kan fibration $\rho\colon A \twoheadrightarrow \Gamma$ with small fibers is classified by a pullback square*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads, "\rho"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d, two heads, "\upsilon"] \\
\Gamma \arrow[r, "A"'] & \mathcal{U}
\end{tikzcd} \qquad \square$$

2.3. **Univalence.** We now explain the interpretation of the univalence axiom, originally discovered by Voevodsky in his explorations of the model category of simplicial sets, which can be explained as follows:

> The univalence axiom, when interpreted in a model category, is a statement about a "universe object" $\mathcal{U}$, which is fibrant and comes equipped with a fibration $\upsilon\colon \tilde{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ that is generic, in the sense that any fibration with "small fibers" is a pullback of υ. ...In homotopy theory, it would be natural to ask for the stronger property that $\mathcal{U}$ is a classifying space for small fibrations, i.e. that homotopy classes of maps $\Gamma \to \mathcal{U}$ are in bijection with (rather than merely surjecting onto) equivalence classes of small fibrations over Γ. The univalence axiom is a further strengthening of this: it says that the path space of $\mathcal{U}$ is equivalent to the "universal space of equivalences" between fibers of υ ...In particular, therefore, if two pullbacks of υ are equivalent, then their classifying maps are homotopic. [Sh2, 84]

The univalence axiom concerns the space $\mathsf{Eq}(\tilde{\mathcal{U}}) \twoheadrightarrow \mathcal{U}\times\mathcal{U}$ of equivalences defined for the universal fibration $v\colon \tilde{\mathcal{U}} \to \mathcal{U}$. The idea is that a generalized element

$$\begin{tikzcd} & \mathsf{Eq}(\tilde{\mathcal{U}}) \arrow[d, two heads] \\ \Gamma \arrow[ur, dashed, "e"] \arrow[r, "{(A,B)}"'] & \mathcal{U}\times\mathcal{U} \end{tikzcd}$$

encodes a fibered equivalence $e\colon A \simeq B$ between the fibrations over Γ classified by these maps.

We explain how to construct the simplicial set $\mathsf{Eq}(\tilde{\mathcal{U}})$ via a construction that makes sense in any simplicially locally cartesian closed, simplicial model category $\mathcal{E}$ that is right proper and whose cofibrations are the monomorphisms.

The total space of equivalences between spaces in the universe $\mathcal{U}$ is

$$\Sigma_{A,B:\mathcal{U}} A \simeq B$$

where

$$A \simeq B := \Sigma_{f:A\to B}\mathsf{isEquiv}(f)$$

where

$$\mathsf{isEquiv}(f) := \Sigma_{b:B}\mathsf{isContr}(\mathsf{fib}_f\, b)$$

where

$$\mathsf{isContr}(C) := \Sigma_{c:C}\Pi_{x:C}\, c \sim x$$

and

$$\mathsf{fib}_f\, b := \Sigma_{a:A} f\,a \sim b$$

is the homotopy fiber of f over b. We've introduced each of these constituent constructions already. It remains only to explain how to put them in the context of the universe $\mathcal{U}$.

The context for the space of equivalences $\mathcal{U}\times\mathcal{U}$; in the space $A \simeq B$, constructured in 1.3.13 in an arbitrary context Γ, the A and B can be regarded as a generalized element $(A,B)\colon \Gamma \to \mathcal{U}\times\mathcal{U}$. Projecting to each of the universe factors individually, this map defines a pair of fibrations

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \tilde{\mathcal{U}}\times\mathcal{U} \arrow[r] \arrow[d, two heads, "\pi_1^* v"'] \arrow[dr, phantom, "\lrcorner" very near start] & \tilde{\mathcal{U}} \arrow[d, two heads, "v"] \\ \Gamma \arrow[r, "{(A,B)}"] \arrow[rr, bend right, "A"'] & \mathcal{U}\times\mathcal{U} \arrow[r, "\pi_1"] & \mathcal{U} \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \tilde{\mathcal{U}}\times\mathcal{U} \arrow[r] \arrow[d, two heads, "\pi_2^* v"'] \arrow[dr, phantom, "\lrcorner" very near start] & \tilde{\mathcal{U}} \arrow[d, two heads, "v"] \\ \Gamma \arrow[r, "{(A,B)}"] \arrow[rr, bend right, "B"'] & \mathcal{U}\times\mathcal{U} \arrow[r, "\pi_2"] & \mathcal{U} \end{tikzcd}$$

The internal hom from the former fibration to the latter in the slice over $\mathcal{U}\times\mathcal{U}$ pulls back to define the space, in context Γ, of maps from A to B over Γ, namely the internal hom in the slice $s\mathcal{S}\mathsf{et}_{/\Gamma}$.

$$\begin{tikzcd} \mathsf{map}_\Gamma(A,B) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\tilde{\mathcal{U}}\times\mathcal{U}, \mathcal{U}\times\tilde{\mathcal{U}}) \arrow[d, two heads, "{\mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\pi_1^* v, \pi_2^* v)}"] \\ \Gamma \arrow[r, "{(A,B)}"'] & \mathcal{U}\times\mathcal{U} \end{tikzcd}$$

The fibration $\mathsf{Eq}(\tilde{\mathcal{U}}) \twoheadrightarrow \mathcal{U}\times\mathcal{U}$ is the sum along the fibration $\mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\pi_1^* v, \pi_2^* v)$ of the fibration $\mathsf{isEquiv}(\mu)$ for a map μ in context $\mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\tilde{\mathcal{U}}\times\mathcal{U}, \mathcal{U}\times\tilde{\mathcal{U}})$ that we now construct. The counit ϵ defining the evaluation map for the internal hom pulls back to define a map

$$\mu\colon \mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\tilde{\mathcal{U}}\times\mathcal{U}, \mathcal{U}\times\tilde{\mathcal{U}}) \times_{\mathcal{U}\times\mathcal{U}} (\tilde{\mathcal{U}}\times\mathcal{U}) \to \mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\tilde{\mathcal{U}}\times\mathcal{U}, \mathcal{U}\times\tilde{\mathcal{U}}) \times_{\mathcal{U}\times\mathcal{U}} (\mathcal{U}\times\tilde{\mathcal{U}})$$

over $\mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\tilde{\mathcal{U}}\times\mathcal{U},\mathcal{U}\times\tilde{\mathcal{U}})$. The map μ is the "universal map" in the sense that the fiber of μ over an element $f\colon\Gamma\to\mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\tilde{\mathcal{U}}\times\mathcal{U},\mathcal{U}\times\tilde{\mathcal{U}})$ over $(A,B)\colon\Gamma\to\mathcal{U}\times\mathcal{U}$ is a map $f\colon A\to B$ over Γ.

$$
\begin{tikzcd}
A \arrow[rr, dotted] \arrow[dr, dotted, "f"] \arrow[dd, dotted, two heads] & & \mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\tilde{\mathcal{U}}\times\mathcal{U},\mathcal{U}\times\tilde{\mathcal{U}})\times_{\mathcal{U}\times\mathcal{U}}(\tilde{\mathcal{U}}\times\mathcal{U}) \arrow[rr, two heads] \arrow[dr, dashed, "\mu"'] \arrow[rrrd, dashed, "\epsilon"] \arrow[dd] & & \tilde{\mathcal{U}}\times\mathcal{U} \arrow[dr, two heads, "\pi_1^*v"'] & \\
& B \arrow[rr, dotted] \arrow[dl, dotted, two heads] & & \mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\tilde{\mathcal{U}}\times\mathcal{U},\mathcal{U}\times\tilde{\mathcal{U}})\times_{\mathcal{U}\times\mathcal{U}}(\mathcal{U}\times\tilde{\mathcal{U}}) \arrow[dl] \arrow[rr] & & \mathcal{U}\times\tilde{\mathcal{U}} \arrow[d, two heads, "\pi_2^*v"] \\
\Gamma \arrow[rr, "f"] \arrow[rrrrr, bend right, "{(A,B)}"'] & & \mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\tilde{\mathcal{U}}\times\mathcal{U},\mathcal{U}\times\tilde{\mathcal{U}}) \arrow[rrr, two heads] & & & \mathcal{U}\times\mathcal{U}
\end{tikzcd}
$$

Construction 2.3.1 (the space of equivalences)**.** We define a fibration

$$
\begin{tikzcd}
A\simeq B \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Eq}(\mathcal{U}) \arrow[d, two heads, "(A\simeq B)_{A,B:\Gamma}"] \\
\Gamma \arrow[r, "{(A,B)}"'] & \mathcal{U}\times\mathcal{U}
\end{tikzcd}
$$

to be

$$\mathsf{Eq}(\tilde{\mathcal{U}}) := \Sigma_{\mathcal{U}\times\mathcal{U}}\Sigma_{\mathsf{map}_{\mathcal{U}\times\mathcal{U}}(\pi_1^*v,\pi_2^*v)}\mathsf{isEquiv}(\mu).$$

In particular, there is a lift

$$
\begin{tikzcd}
& \mathsf{Eq}(\tilde{\mathcal{U}}) \arrow[d, two heads] \\
\mathcal{U} \arrow[ur, dashed, "\mathrm{id}"] \arrow[r, "{(1,1)}"'] & \mathcal{U}\times\mathcal{U}
\end{tikzcd}
$$

classifying the identity equivalence $\tilde{\mathcal{U}}\simeq\tilde{\mathcal{U}}$ over $\mathcal{U}$. We may now state Voevodksy's univalence axiom.

Definition 2.3.2 (Voevodsky [KL])**.** The fibration $v\colon\tilde{\mathcal{U}}\to\mathcal{U}$ is *univalent* if the comparison map from the path space to the space of equivalences defined by path induction is an equivalence:

$$
\begin{tikzcd}
\mathcal{U} \arrow[r, "\mathrm{id}"] \arrow[d, tail, "\mathrm{refl}"', "\wr"] & \mathsf{Eq}(\tilde{\mathcal{U}}) \arrow[d, two heads, "(A\simeq B)_{A,B:\mathcal{U}}"] \\
\mathcal{U}^{\Delta^1} \arrow[ur, dashed, "\text{path-to-eq}"] \arrow[r, two heads, "(A\sim B)_{A,B:\mathcal{U}}"'] & \mathcal{U}\times\mathcal{U}
\end{tikzcd}
\qquad \text{path-to-eq} := \text{path-ind}(\mathrm{refl}\mapsto\mathrm{id})
$$

By the 2-of-3 property, path-to-eq$(\mathrm{refl}\mapsto\mathrm{id})\colon\mathcal{U}^{\Delta^1}\to\mathsf{Eq}(\tilde{\mathcal{U}})$ is a weak equivalence if and only if $\mathrm{id}\colon\mathcal{U}\rightarrowtail\mathsf{Eq}(\tilde{\mathcal{U}})$ is a trivial cofibration, which is the case if and only if either projection $\mathsf{Eq}(\tilde{\mathcal{U}})\twoheadrightarrow\mathcal{U}$ is a trivial fibration:

$$
\begin{tikzcd}
\Delta \arrow[r, "e"] \arrow[d, tail, "i"'] & \mathsf{Eq}(\tilde{\mathcal{U}}) \arrow[d, two heads, "(\Sigma_{A:\mathcal{U}}A\simeq B)_{B:\mathcal{U}}"] \\
\Gamma \arrow[ur, dashed] \arrow[r, "B"'] & \mathcal{U}
\end{tikzcd}
\tag{2.3.3}
$$

This can either be proved directly [KLV, KL] or deduced from the fibrancy of the universe $\mathcal{U}$ by an argument due to Stenzel [Ste, 2.4.3]. Using realignment, both properties can be re-expressed as properties of the model category itself that do not refer explicitly to the universal Kan fibration. In the literature, univalence of the universal fibration is expressed by the *equivalence extension property*, while the fibrancy of the universe becomes the *fibration extension property* [KL, Sa, Sh2].

3. Computing with univalence

Once the univalence axiom is established, we have an equivalence of path spaces on the universe:

$$\begin{tikzcd} \mathcal{U} \arrow[r, tail, "\mathrm{id}", "\sim"'] \arrow[d, tail, "\mathrm{refl}"', "\wr"] & \mathsf{Eq}(\tilde{\mathcal{U}}) \arrow[d, two heads, "(A\simeq B)_{A,B:\mathcal{U}}"] \\ \mathcal{U}^{\Delta^1} \arrow[ur, dashed, "\text{path-to-eq}"] \arrow[r, two heads, "(A\sim B)_{A,B:\mathcal{U}}"'] & \mathcal{U}\times\mathcal{U} \end{tikzcd}$$

defining an inverse equivalence eq-to-path: $\mathsf{Eq}(\tilde{\mathcal{U}}) \xrightarrow{\sim} \mathcal{U}^{\Delta^1}$. Unfortunately, Voevodsky's proof of the univalence axiom uses classical reasoning, which makes the inverse equivalence inexplicit.

Voevodsky's original construction of the universal Kan fibration makes heavy use of the axiom of choice, well-ordering the fibers of a small Kan fibration to destroy their symmetries. This can be avoided through the use of Hofmann-Streicher universes as we present here. However, the law of excluded middle is very intimately baked into the theory of Kan complexes, used to case split between degenerate and non-degenerate simplices in proving that Kan complexes are closed under exponentiation for instance [BCP]. This can be avoided by replacing Kan fibrations by *uniform Kan fibrations*, with chosen fillers for horns satisfying certain coherence conditions [GS], but then the corresponding classifying universes fail to exist, unless one also restricts the cofibrant objects to be those with decidable degeneracies [GH].

Alternatively, various collaborations have discovered that it is possible to give a fully constructive verification of the univalence axiom by replacing the indexing category $\mathbb{\Delta}$ but a suitably-chosen category of *cubical sets* using for instance:

- the *symmetric cube category* with faces, degeneracies, and (dimension-permuting) symmetries [BCH];
- the *cartesian cube category* with faces, degeneracies, symmetries, and diagonals [ABCFHL, A1];
- the *Dedekind cube category* with faces, degeneracies, symmetries, diagonals, and connections [CCHM]; or
- the *deMorgan cube category* with faces, degeneracies, symmetries, diagonals, connections, and reversals [CCHM].

In each of these categories the 1-cube I^1, which replaces the simplicial interval Δ^1 in the definition of path spaces, is *tiny*, meaning exponentiation with I^1 has a right adjoint. This is used to give an internal characterization of fibration structures and define the internal universe [LOPS].

Unfortunately, the model structures that are used to endow each category with a cubical notion of Kan fibration and Kan complexes are provably not Quillen equivalent to spaces (with the exception of the case of the Dedekind cubes, which is open). In forthcoming joint work with Steve Awodey, Evan Cavallo, Thierry Coquand, and Christian Sattler, we build a Quillen equivalent model structure on the category of presheaves over the cartesian cube category whose fibrations satisfy an addition *equivariance* condition and show this gives the sought-for constructive model of homotopy type theory that presents classical homotopy theory.

The cartesian cube category permits an enhanced version of the transport operation of Construction 1.1.7 when a cubical Kan fibration is defined not in the usual way — by lifting against pushout products of monomorphisms with the endpoint inclusions in the 1-cube I^1 — but by lifting against pushout products of monomorphisms with the inclusion of the "generic point," meaning the element defined by the diagonal morphism $\delta\colon I^1 \to I^1 \times I^1$ in context I^1. This is how the Kan fibrations are defined in [ABCFHL].

The reason the model structure of [ABCFHL, A1] fails to be Quillen equivalent to spaces is that quotients of cubes by the dimension-permuting symmetries fail to be weakly contractible. Our idea is to add equivariance conditions to the Kan fibrations by inserting these symmetries as morphisms to define the generating category of trivial cofibrations. Colimits of trivial cofibrations and generating morphisms between them then inherit a canonical trivial cofibration structure, restoring weak contractibility of these cube quotients.

In constructing a model categorical model of homotopy type theory, it is convenient to have a suitable interval object to define the fibered path space factorizations. Because the 1-cube in the category of cartesian cubical sets fails to satisfy the required conditions, we instead start by building a model of homotopy type theory in the category of symmetric sequences of cartesian cubical sets. There we have a canonical interval object $\mathbb{I} := (I^k)_{k:\mathbb{N}}$, namely the symmetric sequence of cubes in each positive dimension with the regular Σ_k-action on the k-cube I^k. The fibrations and trivial fibrations — but not the weak equivalences — in this model structure are then lifted along the constant functor to the category of cartesian cubical sets. Full details will be forthcoming in a paper, which will hopefully appear soon.

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