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Elements of ∞-Category Theory

joint with Dominic Verity

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What are ∞-categories and what are they for?

> *It frames a possible template for any mathematical theory: the theory should have* nouns *and* verbs*, i.e., objects, and morphisms, and there should be an explicit notion of composition related to the morphisms; the theory should, in brief, be packaged by a category.*
>
> —*Barry Mazur, "When is one thing equal to some other thing?"*

An ∞-category frames a template with nouns, verbs, adjectives, adverbs, pronouns, prepositions, conjunctions, interjections,… which has:

- objects $\bullet$ and 1-morphisms between them $\bullet \longrightarrow \bullet$
- composition witnessed by invertible 2-morphisms

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] \arrow[d, phantom, "\wr\|\alpha"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, dashed, "g\circ f"'] & {} & \bullet \end{tikzcd}$$

- associativity

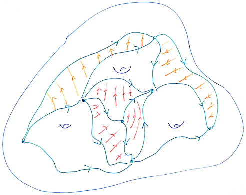
$$\begin{tikzcd} \bullet \arrow[rr, dashed, "h\circ g\circ f"] \arrow[dr, "f"] \arrow[ddr, dashed, "g\circ f"'] & \arrow[d, phantom, "\wr\|\gamma"] & \bullet \\ & \bullet \arrow[ur, dashed, "h\circ g"'] \arrow[d, "g"] \arrow[dl, phantom, "\wr\|\alpha"] \arrow[dr, phantom, "\wr\|\beta"] & \\ & \bullet \arrow[uur, "h"'] & \end{tikzcd}$$

witnessed by invertible 3-morphisms

with these witnesses coherent up to invertible morphisms all the way up.

Examples of ∞-categories

The full homotopy type of a topological space is captured by its fundamental ∞-groupoid whose



- objects are points, 1-morphisms are paths,
- 2-morphisms are homotopies between paths,
- 3-morphisms are homotopies between homotopies, …

The quotient homotopy category recovers the fundamental groupoid of points and based homotopy classes of paths. Similarly:

- The derived category of a ring is the homotopy category of the ∞-category of chain complexes.
- The category of closed n-manifolds and diffeomorphism classes of cobordisms is the homotopy category of the ∞-category of closed n-manifolds and cobordisms.

Here "∞-category" is a nickname for the $n = 1$ special case of an (∞, n)-category, a weak infinite dimensional category in which all morphisms above dimension n are invertible (for fixed $0 \leq n \leq \infty$).

Curiosity 1: a postponed definition of an ∞-category

In the Introductory Workshop for the Derived Algebraic Geometry and Birational Geometry and Moduli Spaces programs at MSRI in February 2019, Carlos Simpson gave a beautiful three-hour lecture course "∞-categories and why they are useful":

> *Abstract: In this series, we'll introduce ∞-categories and explain their relationships with triangulated categories, dg-categories, and Quillen model categories. We'll explain how the ∞-categorical language makes it possible to create a moduli framework for objects that have some kind of up-to-homotopy aspect: stacks, complexes, as well as higher categories themselves. The main concepts from usual category theory generalize very naturally. Emphasis will be given to how these techniques apply in algebraic geometry. In the last talk we'll discuss current work related to mirror symmetry and symplectic geometry via the notion of stability condition.*

What's curious is that a definition of an ∞-category doesn't appear until the second half of the second talk.

Curiosity 2: competing models of ∞-categories

That definition of ∞-categories is used in

- André Hirschowitz, Carlos Simpson — Descente pour les n-champs, 1998.

However a different definition appears in

- Pedro Boavida de Brito, Michael Weiss — Spaces of smooth embeddings and configuration categories, 2018.

yet another definition appears in

- Andrew Blumberg, David Gepner, Gonçalo Tabuada — A universal characterization of higher algebraic K-theory, 2013

and still another definition is used at various points in

- Jacob Lurie — Higher Topos Theory, 2009.

These competing definitions are referred to as models of ∞-categories.

Curiosity 3: the necessity of repetition?

Considerable work has gone into defining the key notions for and proving the fundamental results about ∞-categories, but sometimes this work is later redeveloped starting from a different model.
— e.g., David Kazhdan, Yakov Varshavsky's Yoneda Lemma for Complete Segal Spaces begins:

> *In recent years ∞-categories or, more formally, $(\infty,1)$-categories appear in various areas of mathematics. For example, they became a necessary ingredient in the geometric Langlands problem. In his books [Lu1, Lu2] Lurie developed a theory of ∞-categories in the language of quasi-categories and extended many results of the ordinary category theory to this setting.*
>
> *In his work [Re1] Rezk introduced another model of ∞-categories, which he called complete Segal spaces. This model has certain advantages. For example, it has a generalization to (∞, n)-categories (see [Re2]).*
>
> *It is natural to extend results of the ordinary category theory to the setting of complete Segal spaces. In this note we do this for the Yoneda lemma.*

Curiosity 4: avoiding a precise definition at all

The precursor to Jacob Lurie's Higher Topos Theory is a 2003 preprint On ∞-Topoi, which avoids selecting a model of ∞-categories at all:

> *We will begin in §1 with an informal review of the theory of ∞-categories. There are many approaches to the foundation of this subject, each having its own particular merits and demerits. Rather than single out one of those foundations here, we shall attempt to explain the ideas involved and how to work with them. The hope is that this will render this paper readable to a wider audience, while experts will be able to fill in the details missing from our exposition in whatever framework they happen to prefer.*

Reimagining the foundations of ∞-category theory

A main theme from a new book Elements of ∞-Category Theory is that the theory of ∞-categories is model independent.

www.math.jhu.edu/~eriehl/elements.pdf

In more detail:

- Much of the theory of ∞-categories can be developed model-independently, in an axiomatic setting we call an ∞-cosmos.
- Change-of-model functors define biequivalences of ∞-cosmoi, which preserve, reflect, and create ∞-categorical structures.
- Consequently theorems proven both "synthetically" and "analytically" transfer between models.
- Moreover there is a formal language for expressing properties about ∞-categories that are independent of a choice of model.

Plan

1. A taste of formal category theory

2. Model-independent foundations of ∞-category theory

3. The model-independence of ∞-category theory

1

A taste of formal category theory

Category theory in context

Theorem. $a \times (b + c) = (a \times b) + (a \times c)$ for natural numbers a, b, c.

By categorification, choose sets A, B, C with cardinalities a, b, c and instead exhibit an isomorphism

$$A \times (B + C) \cong (A \times B) + (A \times C)$$

where $\times$ is the cartesian product and $+$ is the disjoint union. By the Yoneda lemma, instead define a natural bijection between functions

$$A \times (B + C) \to X \quad \leftrightsquigarrow \quad (A \times B) + (A \times C) \to X.$$

Proof (that left adjoints preserve colimits):

$$\begin{array}{c} A \times (B + C) \to X \\ \hline B + C \to X^A \\ \hline (B \to X^A, C \to X^A) \\ \hline (A \times B \to X, A \times C \to X) \\ \hline (A \times B) + (A \times C) \to X \end{array} \qquad \square$$

Left adjoints preserve colimits

The same argument — transposing, applying the universal property of the colimit, transposing, and again applying the universal property of the colimit — proves that left adjoints preserve colimits:

- There is a linear isomorphism $U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W)$ of vector spaces.
- The free group on the set $X + Y$ is the free product of the free groups on the sets X and Y.
- For any bimodule M, the tensor product $M \otimes -$ is right exact.
- For any function $f\colon A \to B$, the inverse image function $f^{-1} : \mathcal{P}B \to \mathcal{P}A$ preserves both unions and intersections, while the direct image function $f_* : \mathcal{P}A \to \mathcal{P}B$ preserves unions.
- ...

By the categorical principle of duality, the same argument also proves that right adjoints preserve limits.

Adjunctions and limits as absolute right liftings

An adjunction consists of:

- categories A and B, functors $u \colon A \to B$ and $f \colon B \to A$, and
- a natural transformation $\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] & A \end{tikzcd}$ that is an absolute right lifting:

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] \arrow[r, phantom, shift right=3, "\exists!\Downarrow\beta"] & B \arrow[d, "f"] \\ A \arrow[ur, dashed, "u"] \arrow[r, equal] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] & A \end{tikzcd} \qquad fb \stackrel{\alpha}{\Longrightarrow} a \iff b \stackrel{\beta}{\Longrightarrow} ua$$

A limit of a diagram $d \colon J \to A$ is an absolute right lifting $\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\lambda"] & A^J \end{tikzcd}$

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\gamma"] & A \arrow[d, "\Delta"] \\ 1 \arrow[r, "d"'] & A^J \end{tikzcd} = \begin{tikzcd} X \arrow[r, "a"] \arrow[d, "!"'] \arrow[r, phantom, shift right=3, "\exists!\Downarrow\delta"] & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, dashed, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\lambda"] & A^J \end{tikzcd} \qquad \Delta a \stackrel{\gamma}{\Longrightarrow} d \iff a \stackrel{\delta}{\Longrightarrow} \ell$$

Right adjoints preserve limits

Theorem. Right adjoints preserve limits.

Proof: To see that

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] & A^J \arrow[r, "u^J"'] \arrow[u, phantom, shift right=3, "\Downarrow\lambda"] & B^J \end{tikzcd}$$

is absolute right lifting note

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] & A^J \arrow[r, "u^J"] \arrow[dr, equal] \arrow[u, phantom, shift right=3, "\Downarrow\lambda"] & B^J \arrow[d, "f^J"] \arrow[dl, phantom, "\Downarrow\epsilon^J"] \\ & & A^J \end{tikzcd} = \begin{tikzcd} & & B \arrow[d, "f"] \\ & A \arrow[ur, "u"] \arrow[r, equal] \arrow[d, "\Delta"] \arrow[ur, phantom, shift right=4, "\Downarrow\epsilon"] & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] & A^J \arrow[r, equal] \arrow[u, phantom, shift right=3, "\Downarrow\lambda"] & A^J \end{tikzcd} = \begin{tikzcd} & B \arrow[d, "f"] \\ & A \arrow[d, "\Delta"] \arrow[u, phantom, shift right=4, "\Downarrow\epsilon\ell"] \\ 1 \arrow[uur, bend left, "u\ell"] \arrow[ur, "\ell"] \arrow[r, "d"'] & A^J \arrow[u, phantom, shift right=3, "\Downarrow\lambda"] \end{tikzcd}$$

and that absolute right liftings compose and cancel on the bottom. □

Surprisingly, this same argument proves that right adjoints between ∞-categories preserve ∞-categorical limits.

2

Model-independent foundations of ∞-category theory

Models of ∞-categories

The meaning of the term ∞-category is made precise by several models, connected by "change-of-model" functors.

$$\begin{tikzcd}
 & \mathcal{R}\mathrm{ezk} \arrow[r] \arrow[dr, shift left] & \mathcal{S}\mathrm{egal} \arrow[d, shift left] & \\
\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \arrow[ur] \arrow[dr] \arrow[drr] & & & \mathcal{T}\mathrm{op}\text{-}\mathcal{C}\mathrm{at} \arrow[dl] \\
 & 1\text{-}\mathcal{C}\mathrm{omp} \arrow[r, shift left] & q\mathcal{C}\mathrm{at} \arrow[l, dashed, shift left] \arrow[ul, shift left] \arrow[u, shift left] &
\end{tikzcd}$$

- topological categories and relative categories are the simplest to define but the maps between them are too strict
- $\left\{\begin{array}{l}\text{quasi-categories (née weak Kan complexes),}\\ \text{Rezk spaces (née complete Segal spaces),}\\ \text{Segal categories, and}\\ \text{(saturated 1-trivial weak) 1-complicial sets}\end{array}\right.$
  each have the correct maps and also an internal hom, and in fact any of these categories can be enriched over any of the others

The analytic vs synthetic theory of ∞-categories

Q: How might you develop the category theory of ∞-categories?

Two strategies:

- work analytically to give categorical definitions and prove theorems using the combinatorics of one model

  (eg., Joyal, Lurie, Gepner-Haugseng, Cisinski in $q\mathcal{C}at$; Kazhdan-Varshavsky, Rasekh in $\mathcal{R}ezk$; Simpson in $\mathcal{S}egal$)

- work synthetically to give categorical definitions and prove theorems in all four models $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ at once

Our method: introduce an ∞-cosmos to axiomatize the common features of the categories $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ of ∞-categories.

∞-cosmoi of ∞-categories

Idea: An ∞-cosmos is an infinite-dimensional category whose objects are ∞-categories: an "$(\infty, 2)$-category with $(\infty, 2)$-categorical limits."

An ∞-cosmos is a category that

- is enriched over quasi-categories, i.e., functors $f\colon A \to B$ between ∞-categories define the points of a quasi-category $\mathrm{Fun}(A, B)$,
- has a class of isofibrations $E \twoheadrightarrow B$ with familiar closure properties,
- and has the expected limits of diagrams of ∞-categories and isofibrations, which satisfy simplicially-enriched universal properties.

Theorem. $\mathcal{qC}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, and $1\text{-}\mathcal{C}omp$ define ∞-cosmoi, and so do certain models of (∞, n)-categories for $0 \leq n \leq \infty$, fibered versions of all of the above, and many more things besides.

Henceforth ∞-category and ∞-functor are technical terms that refer to the objects and morphisms of some ∞-cosmos.

The homotopy 2-category

The homotopy 2-category of an ∞-cosmos is a strict 2-category whose:

- objects are the ∞-categories A, B in the ∞-cosmos
- 1-cells are the ∞-functors $f\colon A \to B$ in the ∞-cosmos
- 2-cells we call ∞-natural transformations $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$ which are defined to be homotopy classes of 1-simplices in $\mathrm{Fun}(A, B)$

Theorem. Equivalences in the homotopy 2-category

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left, equal] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\Downarrow\cong"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, equal] \arrow[r, bend right, "fg"'] \arrow[r, phantom, "\Downarrow\cong"] & B \end{tikzcd}$$

coincide with equivalences in the ∞-cosmos.

Thus, non-evil 2-categorical definitions are "homotopically correct."

The synthetic theory of ∞-categories

Surprisingly, adjunctions between ∞-categories and limits in an ∞-category can be defined in the homotopy 2-category:

- an adjunction is an absolute right lifting:

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, equal] \arrow[r, phantom, shift left=3, "\Downarrow\epsilon"] & A \end{tikzcd}$$

- a limit is an absolute right lifting:

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\lambda"] & A^J \end{tikzcd}$$

Theorem. Right adjoints preserve limits and left adjoints preserve colimits — and the proof is the one given above !

Q: Where did all the hard work go?

A: The more delicate task is to prove that these synthetic definitions coincide with the definitions previously established in the literature when interpreted in one of the models (but they do).

3

The model-independence of ∞-category theory

Model-independence

$$\begin{tikzcd}
\mathcal{R}ezk \arrow[r] \arrow[dr, shift left] & \mathcal{S}egal \arrow[d, shift left] \\
1\text{-}\mathcal{C}omp \arrow[r, shift left] & q\mathcal{C}at \arrow[ul, shift left] \arrow[u, shift left] \arrow[l, shift left]
\end{tikzcd}$$

$\leftsquigarrow$ cosmological biequivalences between models of $(\infty, 1)$-categories

Model-Independence Theorem. Cosmological biequivalences preserve, reflect, and create ∞-categorical properties and structures.

- The existence of an adjoint to a given functor.
- The existence of a limit for a given diagram.
- The property of a given functor defining a cartesian fibration.
- The representability of modules between ∞-categories.
- The existence of a pointwise Kan extension.

Analytically-proven theorems also transfer along biequivalences:

- Universal properties in $(\infty, 1)$-categories are determined elementwise.

A formal language for model-independent statements

Not every statement about ∞-categories is invariant under equivalence, much less change of model: "this ∞-category has a single object." [a]

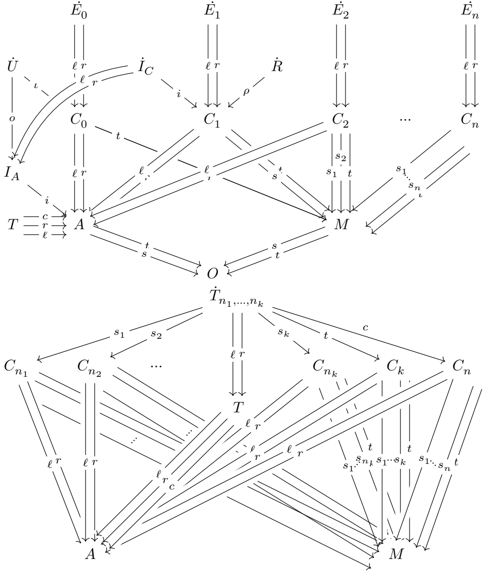
[a] This is related to, but more serious than, the concern raised by Paul Benaceraff in "What numbers could not be": for some constructions of $\mathbb{N}$ "17 has exactly seventeen members" is true, while in others it is false.

Michael Makkai's First-Order Logic with Dependent Sorts is a formal language with restricted equality, suitable for (finite dimensional) higher category theory. This can be adapted to the structure we use to develop the formal theory of ∞-categories in an ∞-cosmos, an extension of the homotopy 2-category called the virtual equipment of modules.

Theorem. Any formula or sentence about ∞-categories written in the FOLDS language of a virtual equipment is invariant under change of model.

A formal language for formal ∞-category theory

The signature for the formal language for model-independent ∞-category theory is a bit complicated:



Summary

- In the past, the theory of ∞-categories has been developed analytically, in a particular model.
- A large part of that theory can be developed simultaneously in many models by working synthetically with ∞-categories as objects in an ∞-cosmos.
- The axioms of an ∞-cosmos are chosen to simplify proofs by allowing us to work strictly up to isomorphism insofar as possible.
- Much of this development in fact takes place in a strict 2-category of ∞-categories, ∞-functors, and ∞-natural transformations using the methods of formal category theory.
- Both analytically- and synthetically-proven results about ∞-categories transfer across "change-of-model" functors called biequivalences.
- Statements about ∞-categories written in a formal language — the language of a virtual equipment — are model-independent.

References

For more on the model-independent theory of ∞-categories see:

Emily Riehl and Dominic Verity

- Elements of ∞-Category Theory, forthcoming from Cambridge University Press

  www.math.jhu.edu/~eriehl/elements.pdf
- ∞-Category Theory from Scratch, mini-course lecture notes:

  Higher Structures, Vol 4, No 1 (2020)
- videos of lecture series given at the Young Topologists' Meeting, the Isaac Newton Institute, and MSRI

Thank you!