Emily Riehl

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Categorifying cardinal arithmetic

Plan

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- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof

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- Step 1: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof
- Epilogue: what was the point of that?

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Step 1: categorification

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A: $a = b$ if and only if A and B are isomorphic, which means there exist functions $f\colon A \to B$ and $g\colon B \to A$ that are inverses in the sense that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. In this case, we write $A \cong B$.

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Categorification: the truth behind $a = b$ is $A \cong B$.

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Q: What is the deeper meaning of the symbols "$+$" and "$\times$"?

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A: The disjoint union $B + C$ is a set with $b + c$ elements.

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A: The cartesian product $A \times B$ is a set with $a \times b$ elements.

$$A = \left\{ \; * \quad \star \; \right\}, \qquad B = \left\{ \begin{matrix} \sharp \\ \flat \\ \natural \end{matrix} \right\}, \qquad A \times B = \left\{ \begin{matrix} (*, \sharp) & (\star, \sharp) \\ (*, \flat) & (\star, \flat) \\ (*, \natural) & (\star, \natural) \end{matrix} \right\}$$

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A: It means that the sets $A \times (B + C)$ and $(A \times B) + (A \times C)$ are isomorphic!

$$A \times (B + C) \cong (A \times B) + (A \times C)$$

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$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

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Step 2: the Yoneda lemma

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to define functions $g\colon B \to A$ and $f\colon A \to B$.

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\begin{tikzcd}
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& \mathrm{Fun}(A,B) \arrow[r, phantom, "\cong"] & \mathrm{Fun}(B,B) & \\
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and similarly $g \circ f = \mathrm{id}_A$. $\square$

Summary of Steps 1 and 2

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$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

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By the Yoneda lemma:

Step 2 summary: To prove $A \times (B + C) \cong (A \times B) + (A \times C)$
$\rightsquigarrow$ we'll instead define a "natural" isomorphism
$$\mathsf{Fun}(A \times (B + C), X) \cong \mathsf{Fun}((A \times B) + (A \times C), X).$$

3

Step 3: representability

The universal property of the disjoint union

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By "pairing"

$$\begin{array}{ccc} \mathsf{Fun}(B+C,X) & \cong & \mathsf{Fun}(B,X)\times\mathsf{Fun}(C,X) \\ \Uparrow\!\!\!\!\in & & \Uparrow\!\!\!\!\in \\ f & \leftrightsquigarrow & (f_B,f_C) \end{array}$$

A universal property of the cartesian product

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4

Step 4: the proof

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5

Epilogue: what was the point of that?

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Exercise: Find a similar proof for other identities of cardinal arithmetic:

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \times \alpha^{\gamma} \quad \text{and} \quad (\alpha^{\beta})^{\gamma} = \alpha^{\beta\times\gamma} = (\alpha^{\gamma})^{\beta}.$$

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$$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W).$$
- For nice topological spaces X, Y, Z,
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- For abelian groups A, B, C,
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Thank you!