EECS208 Discussion 3

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Reading:

- **Appendix A.9 of *High-Dim Data Analysis with Low-Dim Models*.**

1 Matrix Norm

Given a matrix $\boldsymbol{X} \in \mathbb{R}^{m\times n}$, we say $\|\cdot\|_\diamond$ defines a norm function of $\mathbb{R}^{m\times n}$ if $\|\cdot\|_\diamond$ satisfies

- $\|\boldsymbol{X}\|_\diamond \geq 0$, equality holds if and only if $\boldsymbol{X} = \boldsymbol{0}_{m,n}$
- $\|\alpha\boldsymbol{X}\|_\diamond = |\alpha| \|\boldsymbol{X}\|_\diamond$
- $\|\boldsymbol{X} + \boldsymbol{Y}\|_\diamond \leq \|\boldsymbol{X}\|_\diamond + \|\boldsymbol{Y}\|_\diamond$.

Some typical matrix norms are

- $\|\boldsymbol{X}\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^m |x_{ij}|$ (maximum column-wise ℓ^1 norm)
- $\|\boldsymbol{X}\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^n |x_{ij}|$ (maximum row-wise ℓ^1 norm)
- $\|\boldsymbol{X}\|_2 = \sigma_1(\boldsymbol{X}) = \max_{\|\boldsymbol{y}\|_2\leq 1} \|\boldsymbol{X}y\|_2$ (maximum singular value)
- $\|\boldsymbol{X}\|_* = \mathrm{tr}(\sqrt{\boldsymbol{X}^\top\boldsymbol{X}}) = \sum_{i=1}^{\min\{m,n\}} \sigma_i(\boldsymbol{X})$ (sum of all singular values)
- $\|\boldsymbol{X}\|_F = \sqrt{\mathrm{tr}(\boldsymbol{X}^\top\boldsymbol{X})} = \sqrt{\sum_{i=1}^m \sum_{j=1}^n x_{ij}^2} = \sqrt{\sum_{i=1}^{\min m,n} \sigma_i^2(\boldsymbol{X})}$ (square root of sum of square of singular values)

2 Matrix Inequalities

Suppose $\boldsymbol{X} \in \mathbb{R}^{m\times n}$ is a matrix with rank r, then

- $\|\boldsymbol{X}\|_2 \leq \|\boldsymbol{X}\|_F \leq \sqrt{r}\,\|\boldsymbol{X}\|_2$
- $\|\boldsymbol{X}\|_F \leq \|\boldsymbol{X}\|_* \leq \sqrt{r}\,\|\boldsymbol{X}\|_F$
- $\|\boldsymbol{A}\boldsymbol{B}\|_F \leq \|\boldsymbol{A}\|_2 \|\boldsymbol{B}\|_F$
- The inequality between equation (3.2.2) and (3.2.3) in the textbook: $\|\boldsymbol{\Delta}\| \leq \|\boldsymbol{\Delta}\|_F < k\,\|\boldsymbol{\Delta}\|_\infty \leq k\mu(\boldsymbol{A})$, where $\boldsymbol{A}_I^\top \boldsymbol{A}_I = \boldsymbol{I} + \boldsymbol{\Delta}$.

3 How to Prepare for a Project

A course project typically consists of the following parts: 1) introduction, 2) formulation, 3) methods, 4) results, and 5) conclusion/discussion.

1. **Introduction:** The background of this problem, in this section you need to answer these following questions:
   - What is this problem about?
   - Why is it interesting?
   - What are the previous attempts/popular methods to tackle this problem?
2. **Formulation:** The formulation of the problem, in this section you need to formulate the problem rigorously. If you are doing a theoretical problem, what is the thing you want to prove? If you are doing an empirical project, what is the task that you want to complete?
3. **Methods:** For a theoretical project, what is the tentative proof you want to do? For an empirical project, what are the methods that you are planning to apply?
4. **Results:** What is the result of your project? For a theoretical project, have you finish the proof (if not, what is the current stage and what have you tried)? For an empirical project, what are the experimental results?
5. **Conclusion/Discussion** What can you conclude from the results? What are the implications of your results?