I/O BEHAVIOR

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CONTROL ENGINEERING WITH PYTHON

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- 🏦 Mines ParisTech, PSL University

SYMBOLS

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🐍 IMPORTS

```
from numpy import *
from numpy.linalg import *
from scipy.linalg import *
from matplotlib.pyplot import *
from mpl_toolkits.mplot3d import *
from scipy.integrate import solve_ivp
```

🐍 STREAMPLOT HELPER

```
def Q(f, xs, ys):
    X, Y = meshgrid(xs, ys)
    v = vectorize
    fx = v(lambda x, y: f([x, y])[0])
    fy = v(lambda x, y: f([x, y])[1])
    return X, Y, fx(X, Y), fy(X, Y)
```

🧭 CONTEXT

1. **System initially at rest.** $x(0) = 0.$
2. **Black box.** The system state $x(t)$ is unknown.
3. **Input/Output (I/O).** The input determines the output:

$$u(t),\, t \geq 0 \;\to\; y(t),\, t \geq 0.$$

The variation of constants method yields

$$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t).$$

🏷️ SIGNALS & CAUSALITY

A **signal** is a time-dependent function

$$x(t) \in \mathbb{R}^n, \; t \in \mathbb{R}.$$

It is **causal** if

$$t < 0 \; \Rightarrow \; x(t) = 0.$$

📝 CONVENTION

In the sequel, we will assume that time-dependent functions defined only for non-negative times

$$x(t),\ t \geq 0$$

are zero for negative times

$$x(t) = 0,\ t < 0.$$

With this convention, they become causal signals.

🏷️ HEAVISIDE FUNCTION

The **Heaviside function** is the causal signal defined by

$$e(t) = \left| \begin{array}{ll} 1 & \text{if } \; t \geq 0, \\ 0 & \text{if } \; t < 0. \end{array} \right.$$

🏷️ Synonym: **(unit) step signal.**

🏷️ IMPULSE RESPONSE

The system **impulse response** is defined by:

$$H(t) = (Ce^{At}B) \times e(t) + D\delta(t) \in \mathbb{R}^{p \times m}$$

📝 NOTES

- the formula is valid for general (**MIMO**) systems.

  🏷️ **MIMO** = multiple-input & multiple-output.

- $\delta(t)$ is the **unit impulse** signal, we'll get back to it (in the meantime, you may assume that $D = 0$).

📝 SISO SYSTEMS

When $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ the system is **SISO**.

🏷️ **SISO** = single-input & single-output.

Then $H(t)$ is a 1×1 matrix.

We identify it with its unique coefficient $h(t)$:

$$H(t) \in \mathbb{R}^{1 \times 1} = [h(t)], \; h(t) \in \mathbb{R}.$$

💎 I/O BEHAVIOR

Let $u(t), x(t), y(t)$ be causal signals such that:

$$\left| \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array} \right., \; t \geq 0 \; \text{ and } \; x(0) = 0.$$

Then

$$y(t) = (H * u)(t) := \int_{-\infty}^{+\infty} H(t-\tau)u(\tau)\, d\tau.$$

🏷️ CONVOLUTION

The operation $*$ is called a **convolution**.

🔍 IMPULSE RESPONSE

Consider the SISO system

$$\left| \begin{array}{rcl} \dot{x} & = & ax + u \\ y & = & x \end{array} \right.$$

where $a \neq 0$.

We have

$$
\begin{aligned}
H(t) &= (Ce^{At}B) \times e(t) + D\delta(t) \\
&= [1]e^{[a]t}[1]e(t) + [0]\delta(t) \\
&= [e(t)e^{at}]
\end{aligned}
$$

When $u(t) = e(t)$ for example,

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} e(t-\tau) e^{a(t-\tau)} e(\tau)\, d\tau \\ &= \int_0^t e^{a(t-\tau)}\, d\tau \\ &= \int_0^t e^{a\tau}\, d\tau \\ &= \frac{1}{a}\left(e^{at} - 1\right) \end{aligned}$$

🧩 INTEGRATOR

Let

$$\left|\begin{array}{rcl} \dot{x} & = & u \\ y & = & x \end{array}\right.$$

where $u \in \mathbb{R}$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

1. 🧮

Compute the impulse response of the system.

🔓 INTEGRATOR

1. 🔓

$$
\begin{aligned}
H(t) &= (Ce^{At}B) \times e(t) + D\delta(t) \\
&= [1]e^{[0]t}[1]e(t) + [0]\delta(t) \\
&= [e(t)]
\end{aligned}
$$

DOUBLE INTEGRATOR

Let

$$\left| \begin{array}{ccc} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & u \\ y & = & x_1 \end{array} \right.$$

where $u \in \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$ and $y \in \mathbb{R}$.

1. 🧮

Compute the impulse response of the system.

🔓 DOUBLE INTEGRATOR

1. 🔓

$$
\begin{aligned}
H(t) &= (C\exp(At)B) \times e(t) + D\delta(t) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} e(t) + [0]\delta(t) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} e(t) \\
&= [te(t)]
\end{aligned}
$$

🧩 GAIN

Let

$$y = Ku$$

where $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $K \in \mathbb{R}^{p \times m}$.

1. 🧮

Compute the impulse response of the system.

🔓 GAIN

1. 🔓

The I/O behavior can be represented by $\dot{x} = 0x + 0u$ and $y = 0 \times x + Ku$ (for example). Thus,

$$
\begin{aligned}
H(t) &= (C \exp(At) B) \times e(t) + D\delta(t) \\
&= 0 + K\delta(t) \\
&= K\delta(t)
\end{aligned}
$$

MIMO SYSTEM

Let

$$H(t) := \begin{bmatrix} e^{t} e(t) & e^{-t} e(t) \end{bmatrix}$$

1. 🧮

Find a linear system with matrices A, B, C, D whose impulse response is $H(t)$.

2. 🧮

Is there another 4-uple of matrices A, B, C, D with the same impulse response?

3. 🧮

Same question but with a matrix A of a different size?

🔓 MIMO SYSTEM

1. 🔓

Since

$$\exp\left(\begin{bmatrix}+1 & 0\\ 0 & -1\end{bmatrix}t\right) = \begin{bmatrix}e^{+t} & 0\\ 0 & e^{-t}\end{bmatrix},$$

the following matrices work:

$$A = \begin{bmatrix}+1 & 0\\ 0 & -1\end{bmatrix},\ B = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ C = \begin{bmatrix}1 & 1\end{bmatrix},\ D = \begin{bmatrix}0 & 0\end{bmatrix}.$$

2. 🔓

Since

$$
\begin{aligned}
H(t) &= (C\exp(At)B) \times e(t) + D\delta(t) \\
&= ((-C)\exp(At)(-B)) \times e(t) + D\delta(t)
\end{aligned}
$$

changing B and C to be

$$
B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \; C = \begin{bmatrix} -1 & -1 \end{bmatrix},
$$

doesn't change the impulse response.

3. 🔓

We can also easily add a scalar dynamics (say $\dot{x}_3 = 0$) that doesn't influence the impulse response.

The following matrices also work

$$A = \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

🏷️ LAPLACE TRANSFORM

Let $x(t), t \in \mathbb{R}$ be a scalar signal.

It **Laplace transform** is the function of s given by:

$$x(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} \, dt.$$

DOMAIN & CODOMAIN

The Laplace transform of a signal is a complex-valued function; its domain is a subset of the complex plane.

$$s \in D \; \Rightarrow \; x(s) \in \mathbb{C}.$$

If $x(t)$ is a causal signal of **sub-exponential growth**

$$|x(t)| \leq k e^{\sigma t} e(t),\; t \in \mathbb{R},$$

($k \geq 0$ and $\sigma \in \mathbb{R}$), its Laplace transform is defined on an open half-plane:

$$\Re(s) > \sigma \;\Rightarrow\; x(s) \in \mathbb{C}.$$

⚠️ NOTATION

We use the same symbol (here “x”) to denote:

- a signal $x(t)$ and
- its Laplace transform $x(s)$

They are two equivalent representations of the same “object”, but different mathematical “functions”.

If you fear some ambiguity, use named variables, e.g.:

$$x(t=1) \;\text{ or }\; x(s=1) \;\text{ instead of }\; x(1).$$

VECTOR/MATRIX-VALUED SIGNALS

The Laplace transform

- of a vector-valued signal $x(t) \in \mathbb{R}^n$ or
- of a matrix-valued signal $X(t) \in \mathbb{R}^{m \times n}$

are computed elementwise.

$$x_i(s) := \int_{-\infty}^{+\infty} x_i(t)e^{-st}\,dt.$$

$$X_{ij}(s) := \int_{-\infty}^{+\infty} X_{ij}(t)e^{-st}\,dt.$$

🏷️ RATIONAL SIGNALS

We will only deal with **rational** (and causal) signals:

$$x(t) = \left(\sum_{\lambda \in \Lambda} p_{\lambda}(t) e^{\lambda t} \right) e(t)$$

where:

- Λ is a finite subset of $\mathbb{C}$,
- for every $\lambda \in \Lambda$, $p_{\lambda}(t)$ is a polynomial in t.

They are called **rational** since

$$x(s) = \frac{n(s)}{d(s)}$$

where $n(s)$ and $d(s)$ are polynomials; also

$$\deg n(s) \leq \deg d(s).$$

🔍 EXPONENTIAL

Let

$$x(t) = e^{at}e(t),\ t \in \mathbb{R}$$

for some $a \in \mathbb{R}$. Then

$$x(s) = \int_{-\infty}^{+\infty} e^{at}e(t)e^{-st}\,dt = \int_{0}^{+\infty} e^{(a-s)t}\,dt.$$

If $\Re(s) > a$, then

$$\left|e^{(a-s)t}\right| \leq e^{-(\Re(s)-a)t};$$

the function $t \in [0, +\infty[\mapsto e^{(a-s)t}$ is integrable and

$$x(s) = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^{+\infty} = \frac{1}{s-a}.$$

💻 SYMBOLIC COMPUTATION

```python
import sympy
from sympy.abc import t, s
from sympy.integrals.transforms \
    import laplace_transform

def L(f):
    return laplace_transform(f, t, s)[0]
```

```
>>> from sympy.abc import a
>>> xt = sympy.exp(a*t)
>>> xs = L(xt)
>>> xs
1/(-a + s)
```

🧩 RAMP

Let

$$x(t) = te(t),\ t \in \mathbb{R}.$$

1. 🧮

Compute analytically the Laplace Transform of $x(t)$.

2. 💻

Compute symbolically the Laplace Transform of $x(t)$.

🔓 RAMP

1. 🔓

$$
\begin{aligned}
x(s) &= \int_{-\infty}^{+\infty} te(t)e^{-st}\,dt \\
&= \int_{0}^{+\infty} te^{-st}\,dt.
\end{aligned}
$$

By integration by parts,

$$\begin{aligned} x(s) &= \left[t\frac{e^{-st}}{-s}\right]_0^{+\infty} - \int_0^{+\infty} \frac{e^{-st}}{-s}\,dt \\ &= \frac{1}{s}\int_0^{+\infty} e^{-st}\,dt \\ &= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^{+\infty} \\ &= \frac{1}{s^2} \end{aligned}$$

2. 🔓

With SymPy, we have accordingly:

```
>>> xt = t
>>> xs = L(xt)
>>> xs
s**(-2)
```

🏷️ TRANSFER FUNCTION

Let $H(t)$ be the impulse response of a system.

Its Laplace transform $H(s)$ is the system **transfer function**.

For LTI systems in standard form,

$$H(s) = C[sI - A]^{-1}B + D.$$

💎 OPERATIONAL CALCULUS

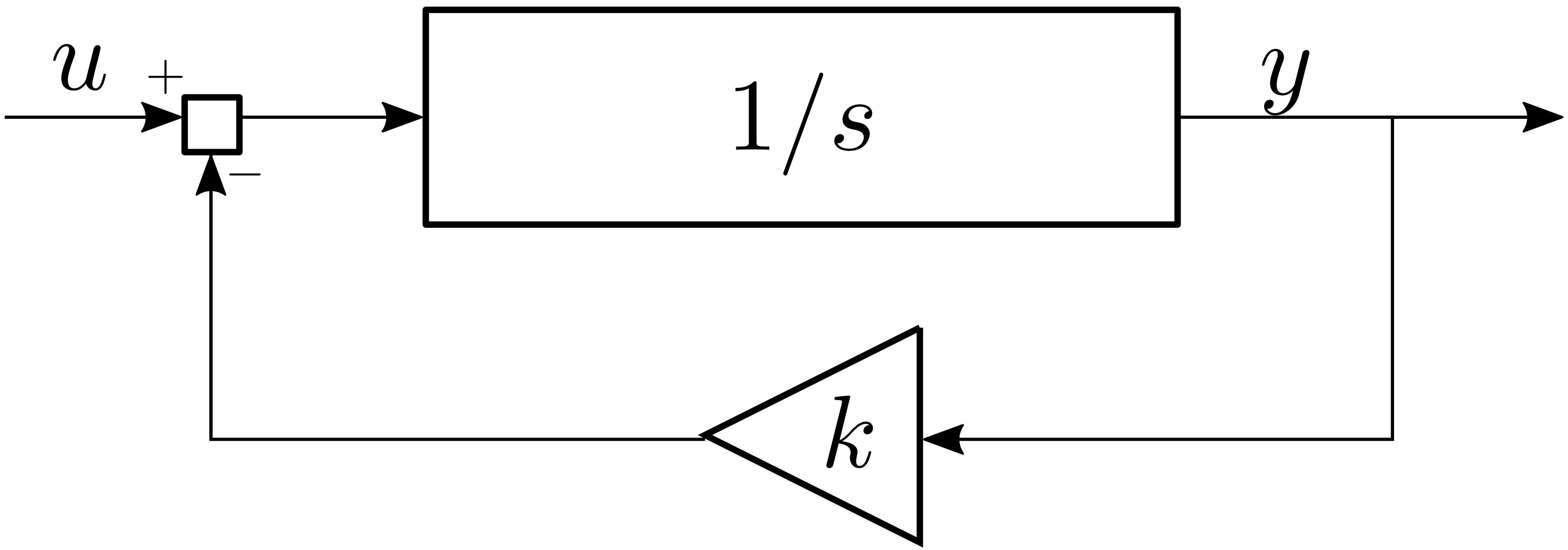
$$y(t) = (H * u)(t) \iff y(s) = H(s) \times u(s)$$

GRAPHICAL LANGUAGE

Control engineers used **block diagrams** to describe (combinations of) dynamical systems, with

- “boxes” to determine the relation between input signals and output signals and
- “wires” to route output signals to inputs signals.

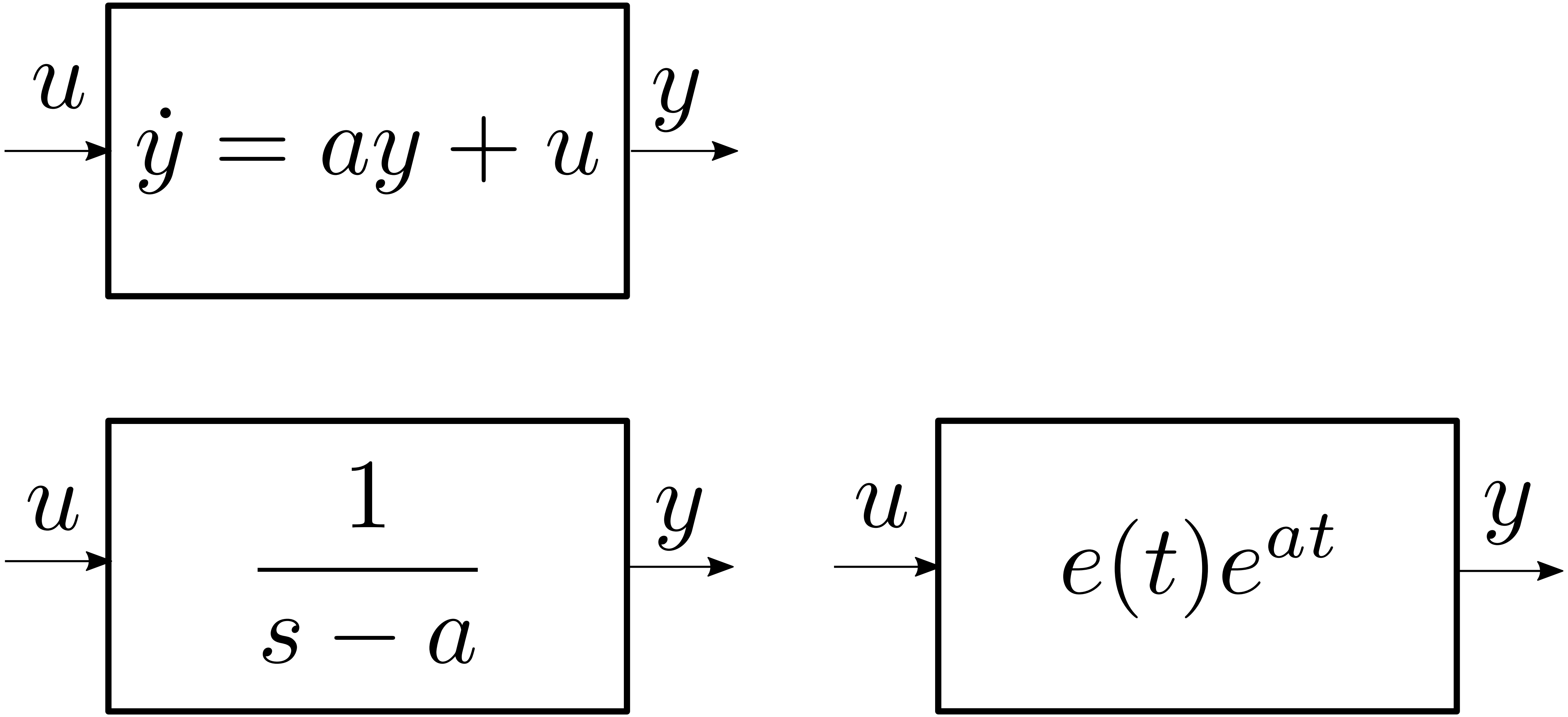
FEEDBACK BLOCK-DIAGRAM



- **Triangles** denote **gains** (scalar or matrix multipliers),
- **Adders** sum (or substract) signals.

- **LTI systems** can be specified by:
  - (differential) equations,
  - the impulse response,
  - the transfer function.

EQUIVALENT SYSTEMS



🧩 FEEDBACK BLOCK-DIAGRAM

Consider the system depicted in the Feedback Block-Diagram picture.

1. 🧮

Compute its transfer function.

🔓 FEEDBACK BLOCK-DIAGRAM

1. 🔓

The diagram logic translates into:

$$y(s) = \frac{1}{s}(u(s) - ky(s)),$$

and thus

$$\left(1 - \frac{k}{s}\right)y(s) = \frac{1}{s}u(s)$$

or equivalently

$$y(s) = \frac{1}{s - k} u(s).$$

Thus, the transfer function of this SISO system is

$$h(s) = \frac{1}{s - k}.$$

🤔 IMPULSE RESPONSE

Why refer to $h(t)$ as the system "impulse response"?

By the way, what's an impulse?

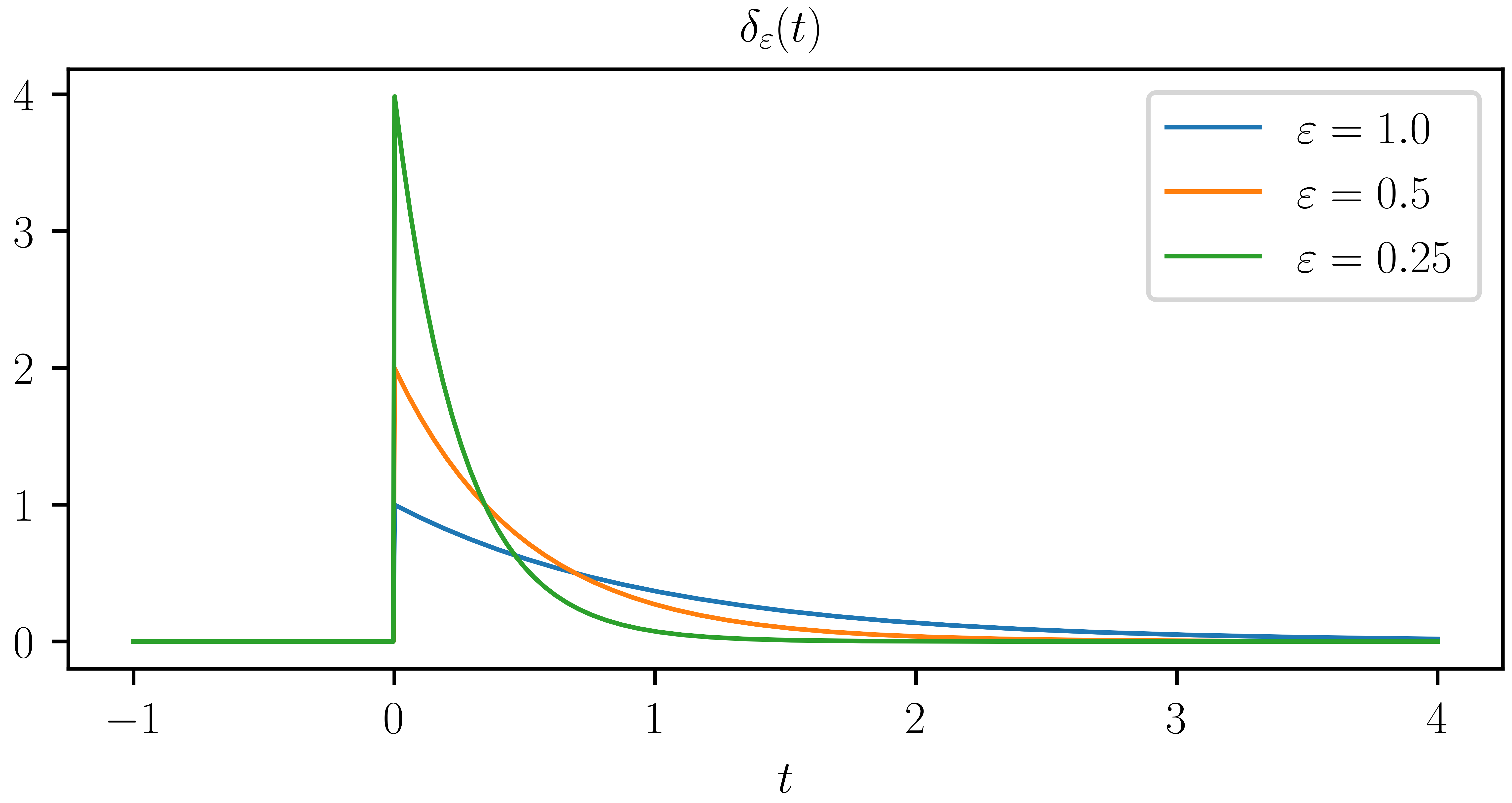
IMPULSE APPROXIMATIONS

Pick a time constant $\varepsilon > 0$ and define

$$\delta_\varepsilon(t) := \frac{1}{\varepsilon} e^{-t/\varepsilon} e(t).$$

```python
def delta(t, eps):
    return exp(-t / eps) / eps * (t >= 0)
```

```python
figure()
t = linspace(-1, 4, 1000)
for eps in [1.0, 0.5, 0.25]:
    plot(t, delta(t, eps),
         label=rf"$\varepsilon={eps}$")
xlabel("$t$"); title(r"$\delta_{\varepsilon}(t)$")
legend()
```



$\delta_\varepsilon(t)$
4
3
2
1
0
$\varepsilon = 1.0$
$\varepsilon = 0.5$
$\varepsilon = 0.25$
−1
0
1
2
3
4
t

IN THE LAPLACE DOMAIN

$$
\begin{aligned}
\delta_\varepsilon(s) &= \int_{-\infty}^{+\infty} \delta_\varepsilon(t) e^{-st}\, dt \\
&= \frac{1}{\varepsilon} \int_{0}^{+\infty} e^{-(s+1/\varepsilon)t}\, dt \\
&= \frac{1}{\varepsilon} \left[\frac{e^{-(s+1/\varepsilon)t}}{-(s+1/\varepsilon)} \right]_0^{+\infty} = \frac{1}{1+\varepsilon s}
\end{aligned}
$$

(assuming that $\Re(s) > -1/\varepsilon$)

- The “limit” of the signal $\delta_\varepsilon(t)$ when $\varepsilon \to 0$ is not defined *as a function* (issue for $t = 0$) but as a **generalized function** $\delta(t)$, the **unit impulse**.
- This technicality can be avoided in the Laplace domain where

$$\delta(s) = \lim_{\varepsilon \to 0} \delta_\varepsilon(s) = \lim_{\varepsilon \to 0} \frac{1}{1 + \varepsilon s} = 1.$$

Thus, if $y(t) = (h * u)(t)$ and

1. $u(t) = \delta(t)$ then
2. $y(s) = h(s) \times \delta(s) = h(s) \times 1 = h(s)$
3. and thus $y(t) = h(t)$.

Conclusion: the impulse response $h(t)$ is the output of the system when the input is the unit impulse $\delta(t)$.

🏷️ I/O STABILITY

A system is **I/O-stable** if there is a $K \geq 0$ such that

$$\|u(t)\| \leq M, \, t \geq 0$$

$$\Rightarrow$$

$$\|y(t)\| \leq KM, \, t \geq 0.$$

🏷️ More precisely, **BIBO-stability** (“bounded input, bounded output”).

🏷️ TRANSFER FUNCTION POLES

A **pole** of the transfer function $H(s)$ is a $s \in \mathbb{C}$ such that for at least one element $H_{ij}(s)$,

$$|H_{ij}(s)| = +\infty.$$

💎 I/O-STABILITY CRITERIA

A system is I/O-stable if and only if all its poles are in the open left-plane, i.e. such that

$$\Re(s) < 0.$$

💎 INTERNAL STABILITY $\Rightarrow$ I/O-STABILITY

If the system $\dot{x} = Ax$ is asymptotically stable, then for any matrices B, C, D of compatible shapes,

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
$$

is I/O-stable.

🔍 FULLY ACTUATED & MEASURED SYSTEM

If $B = I, C = I$ and $D = 0$, that is

$$\dot{x} = Ax + u, \; y = x$$

then $H(s) = [sI - A]^{-1}$.

Therefore, s is a pole of H iff it's an eigenvalue of A.

Thus, in this case, asymptotic stability and I/O-stability are equivalent.

(This equivalence actually holds under much weaker conditions.)