INTERNAL DYNAMICS

👤 Sébastien Boisgérault

CONTROL ENGINEERING WITH PYTHON

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- 🏦 Mines ParisTech, PSL University

SYMBOLS

🐍	Code	🔍	Worked Example
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🏷️	Definition	💻	Numerical Method
💎	Theorem	🧮	Analytical Method
📝	Remark	🧠	Theory
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🐍 IMPORTS

```
from numpy import *
from numpy.linalg import *
from scipy.linalg import *
from matplotlib.pyplot import *
from mpl_toolkits.mplot3d import *
from scipy.integrate import solve_ivp
```

🐍 STREAMPLOT HELPER

```
def Q(f, xs, ys):
    X, Y = meshgrid(xs, ys)
    v = vectorize
    fx = v(lambda x, y: f([x, y])[0])
    fy = v(lambda x, y: f([x, y])[1])
    return X, Y, fx(X, Y), fy(X, Y)
```

🧭

We are interested in the behavior of the solution to

$$\dot{x} = Ax, \;\; x(0) = x_0 \in \mathbb{R}^n$$

First, we study some elementary systems in this class.

SCALAR CASE, REAL-VALUED

$$\dot{x} = ax$$

$$a \in \mathbb{R},\ x(0) = x_0 \in \mathbb{R}.$$

💎 Solution:

$$x(t) = e^{at}x_0$$

🔓 Proof:

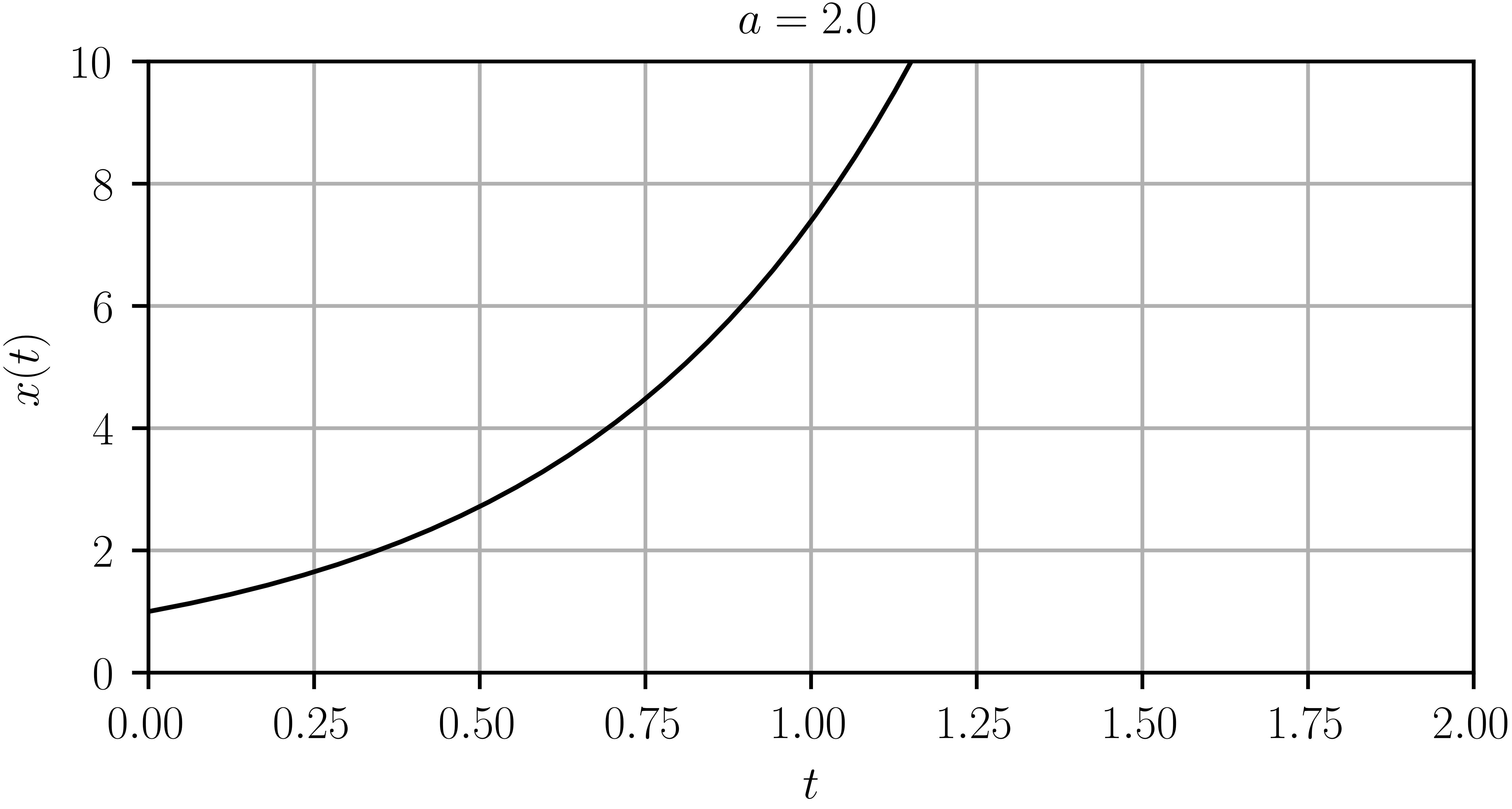
$$\frac{d}{dt}e^{at}x_0 = ae^{at}x_0 = ax(t)$$

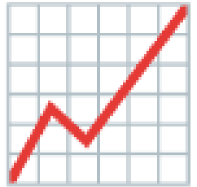
and

$$x(0) = e^{a \times 0}x_0 = x_0.$$

📈 TRAJECTORY

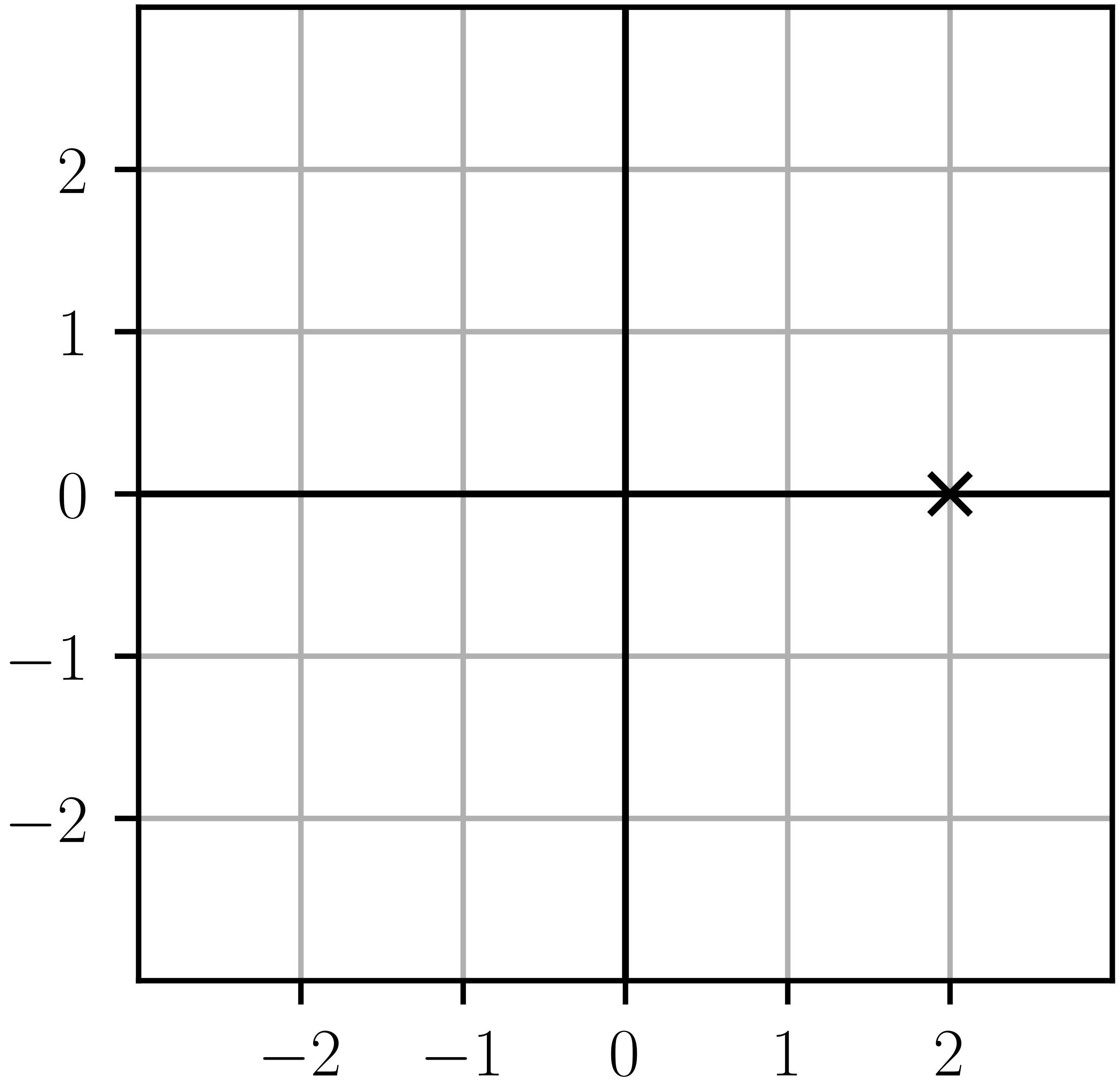
```
a = 2.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```


$a = 2.0$
$x(t)$
0
2
4
6
8
10
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t



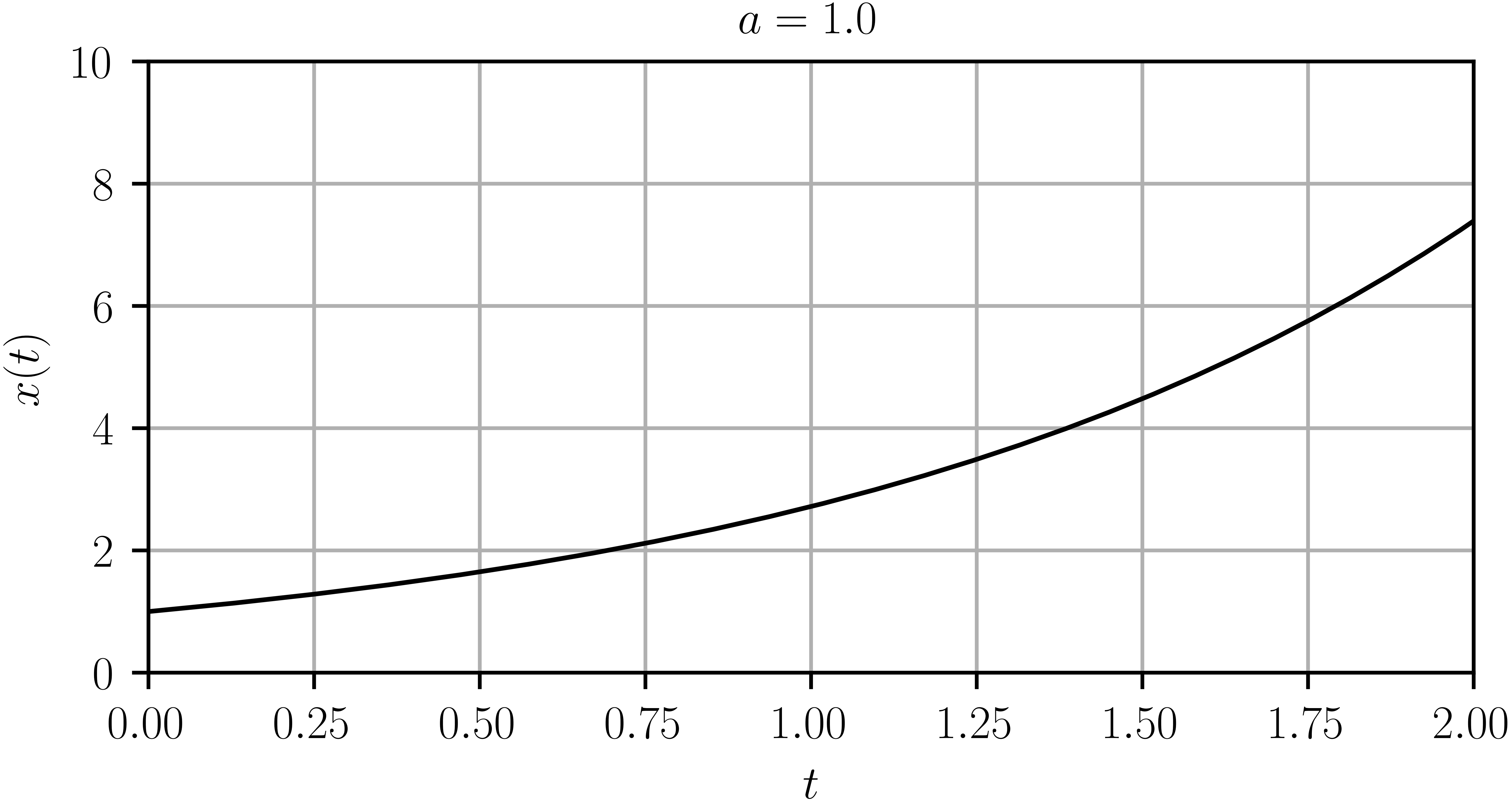
```
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

$a = 2.0$



📈

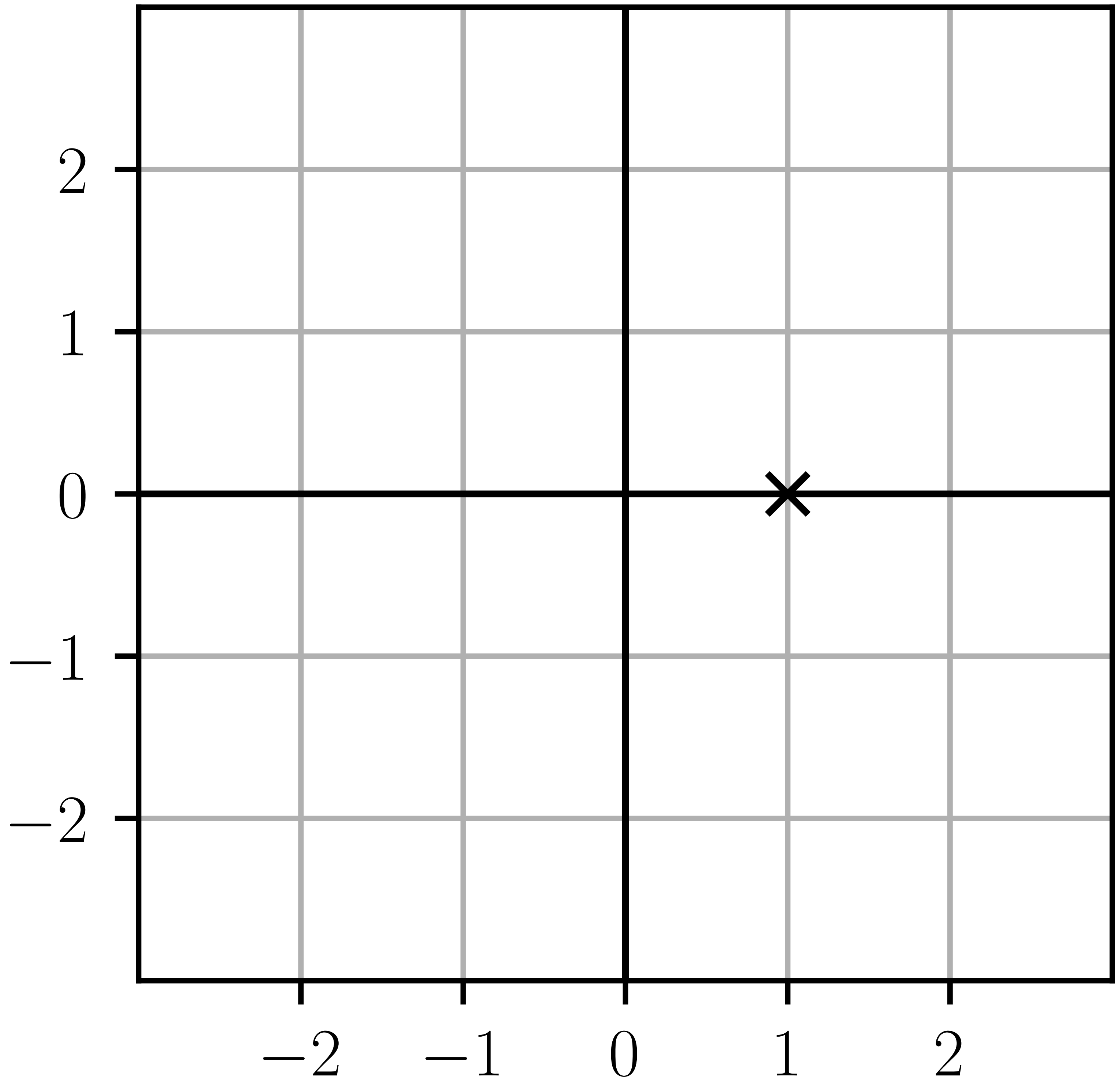
```
a = 1.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

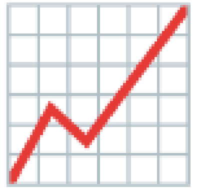


$a = 1.0$
10
8
6
4
2
0
$x(t)$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t

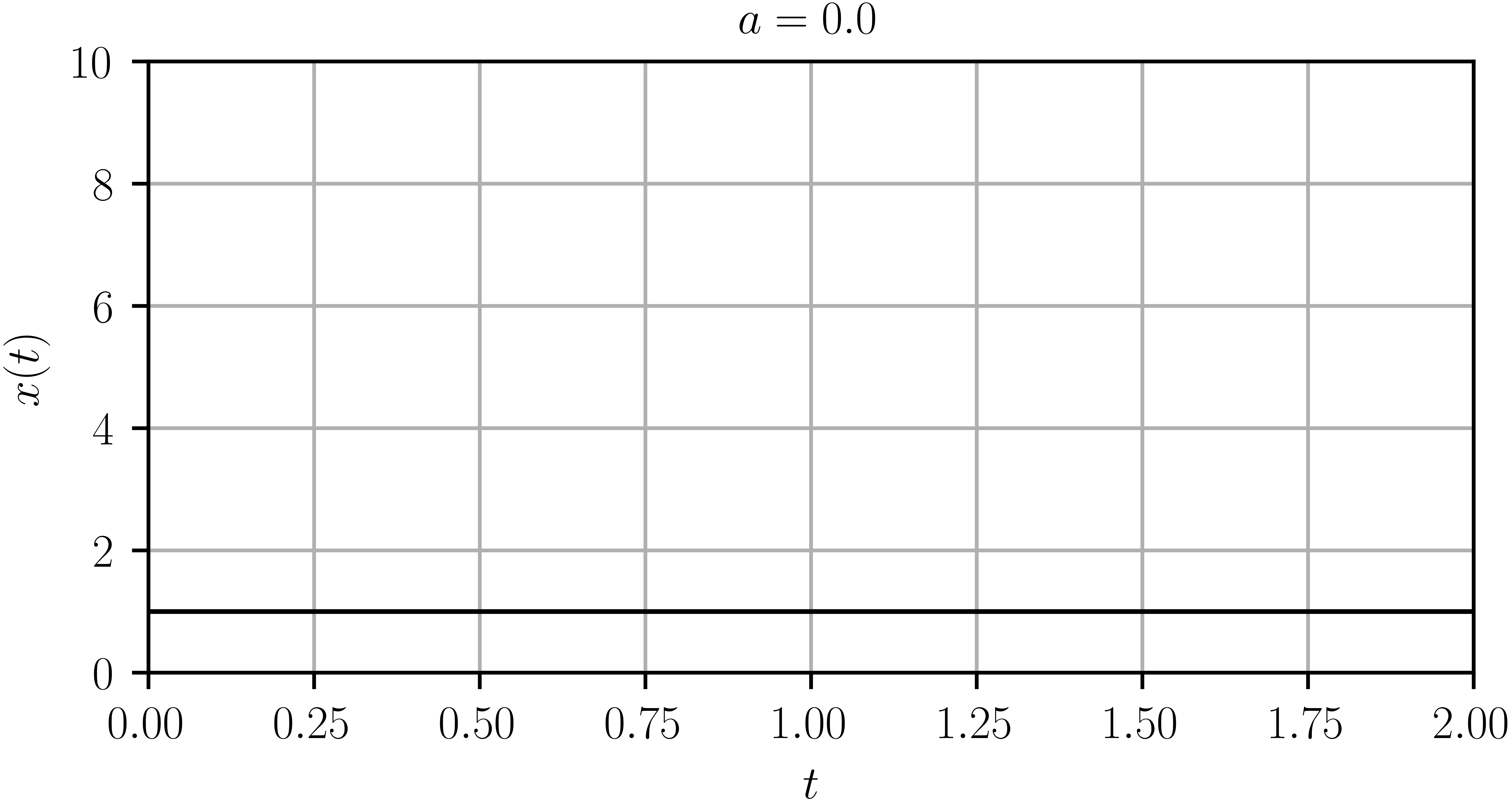
```python
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

$a = 1.0$



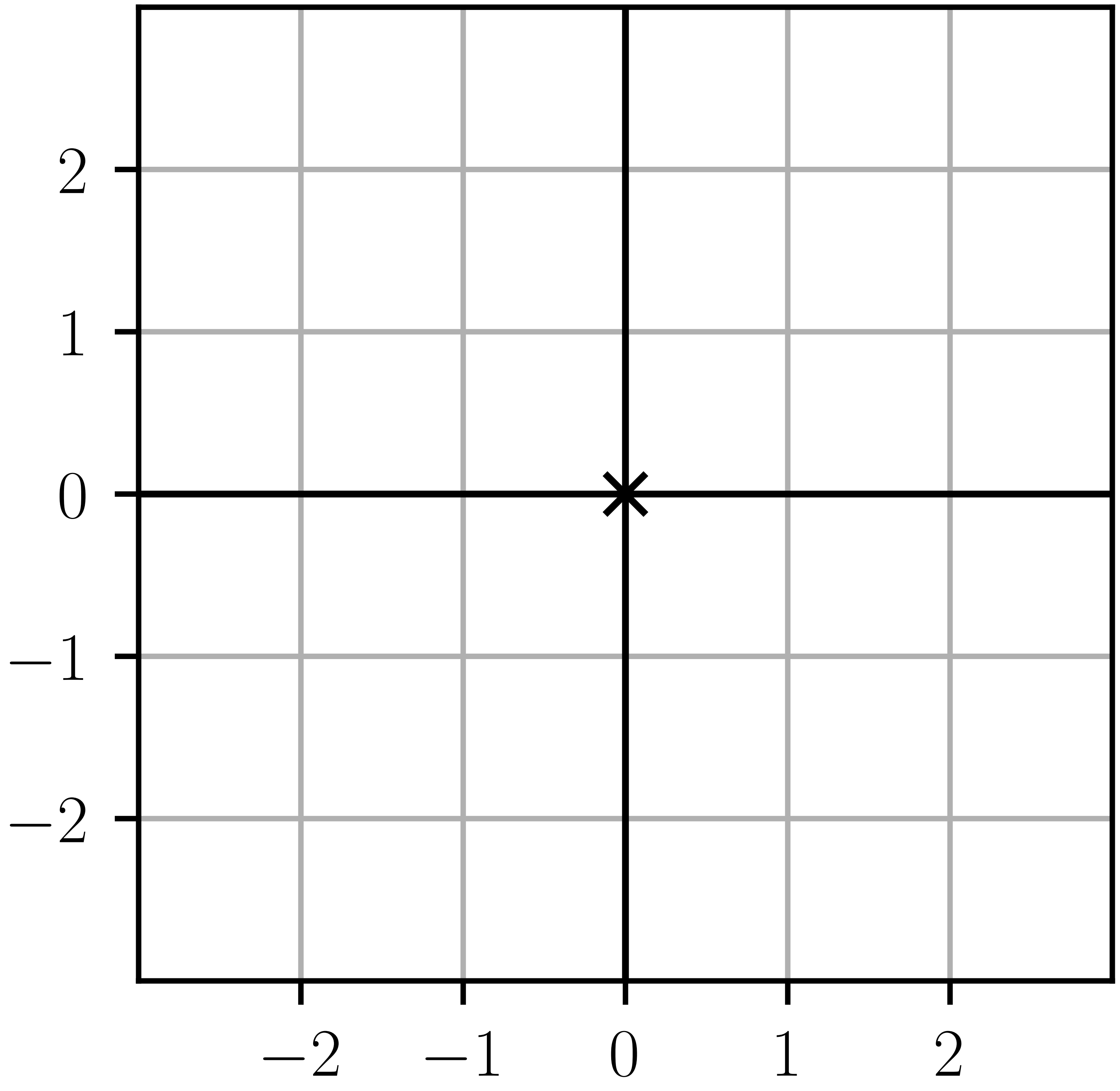


```
a = 0.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```


$a = 0.0$
10
8
6
4
2
0
$x(t)$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t

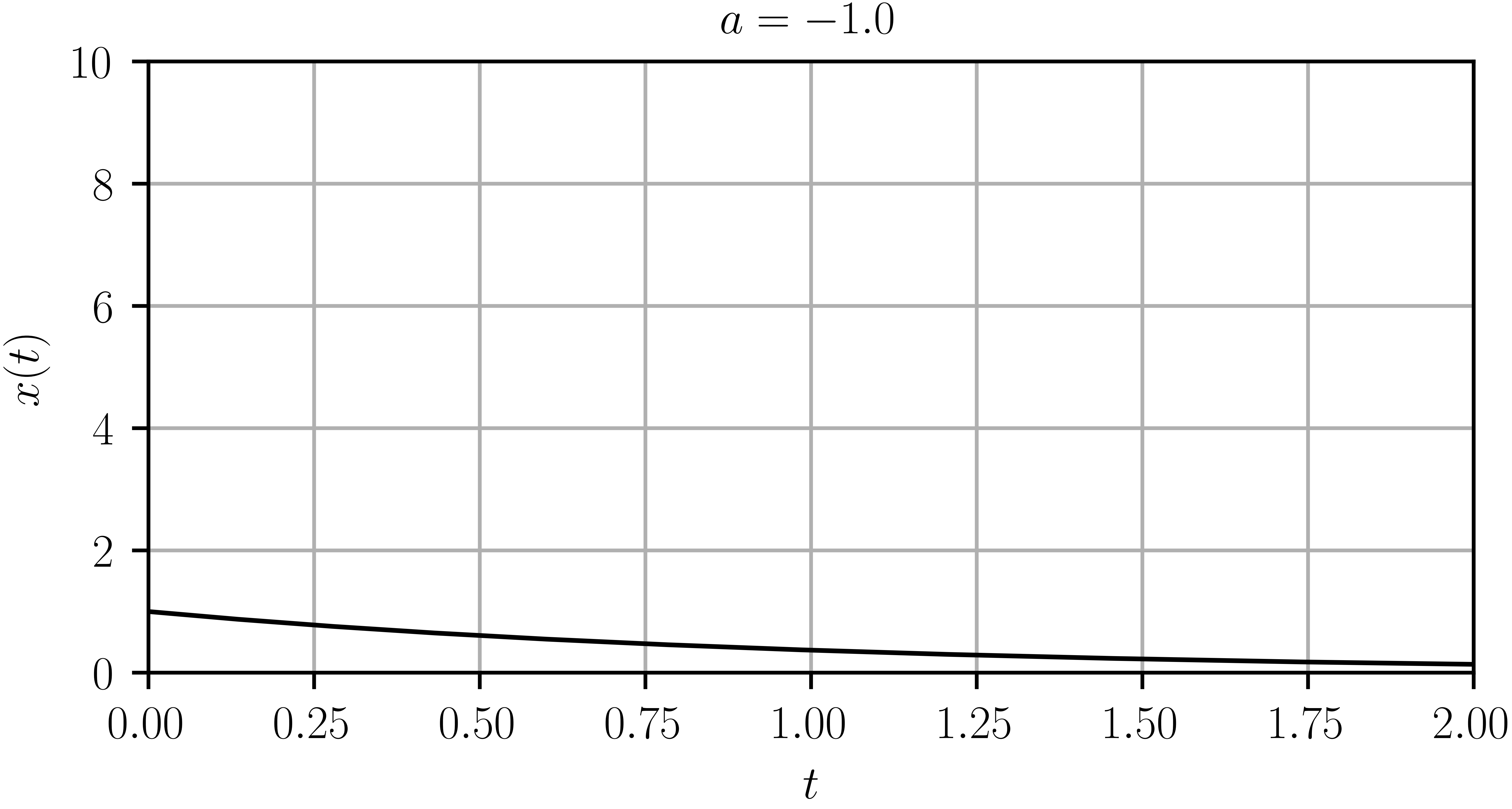
```
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

$a = 0.0$



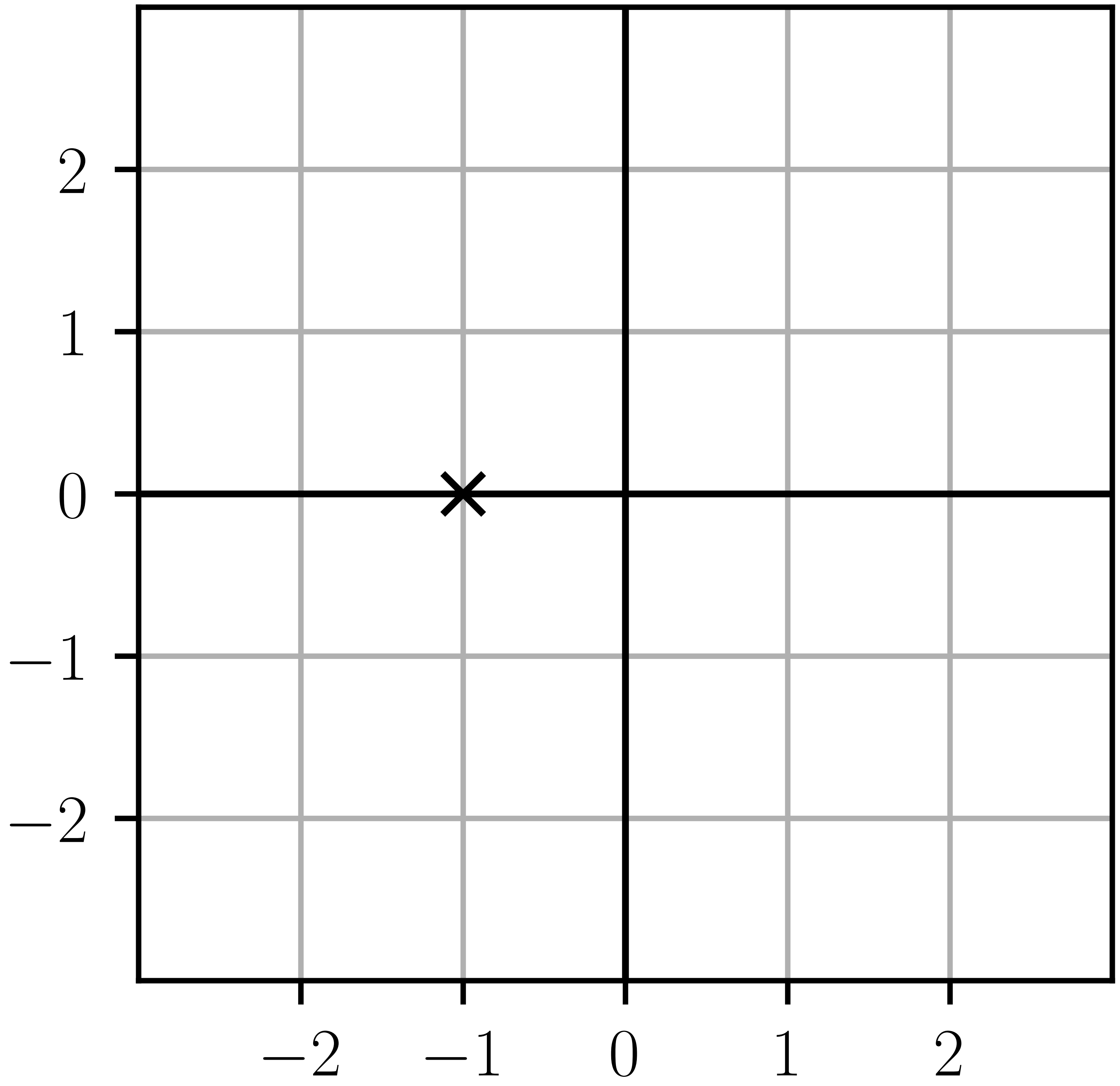
📈

```python
a = -1.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

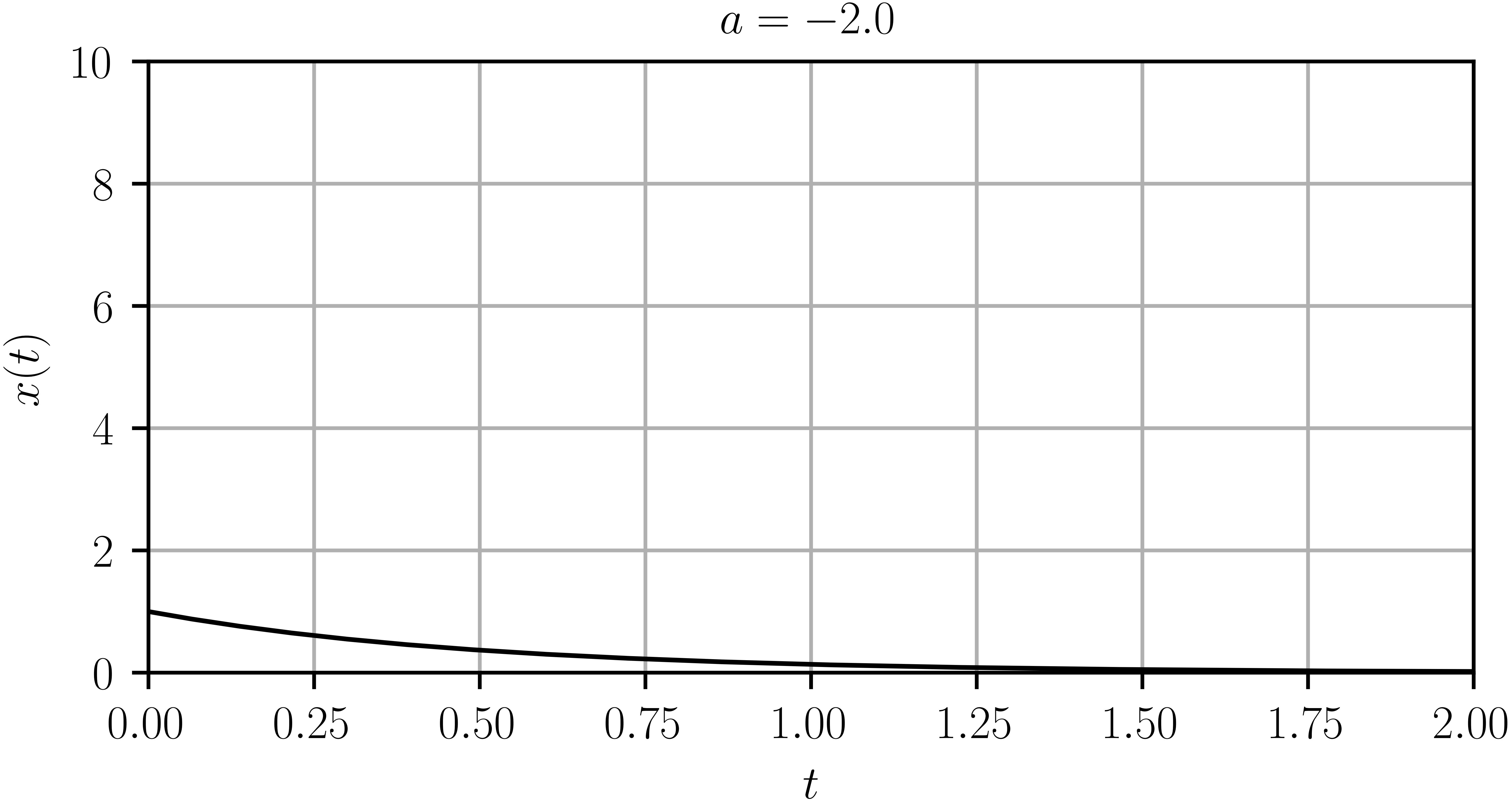

$a = -1.0$
10
8
6
4
2
0
$x(t)$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t

```python
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

$a = -1.0$

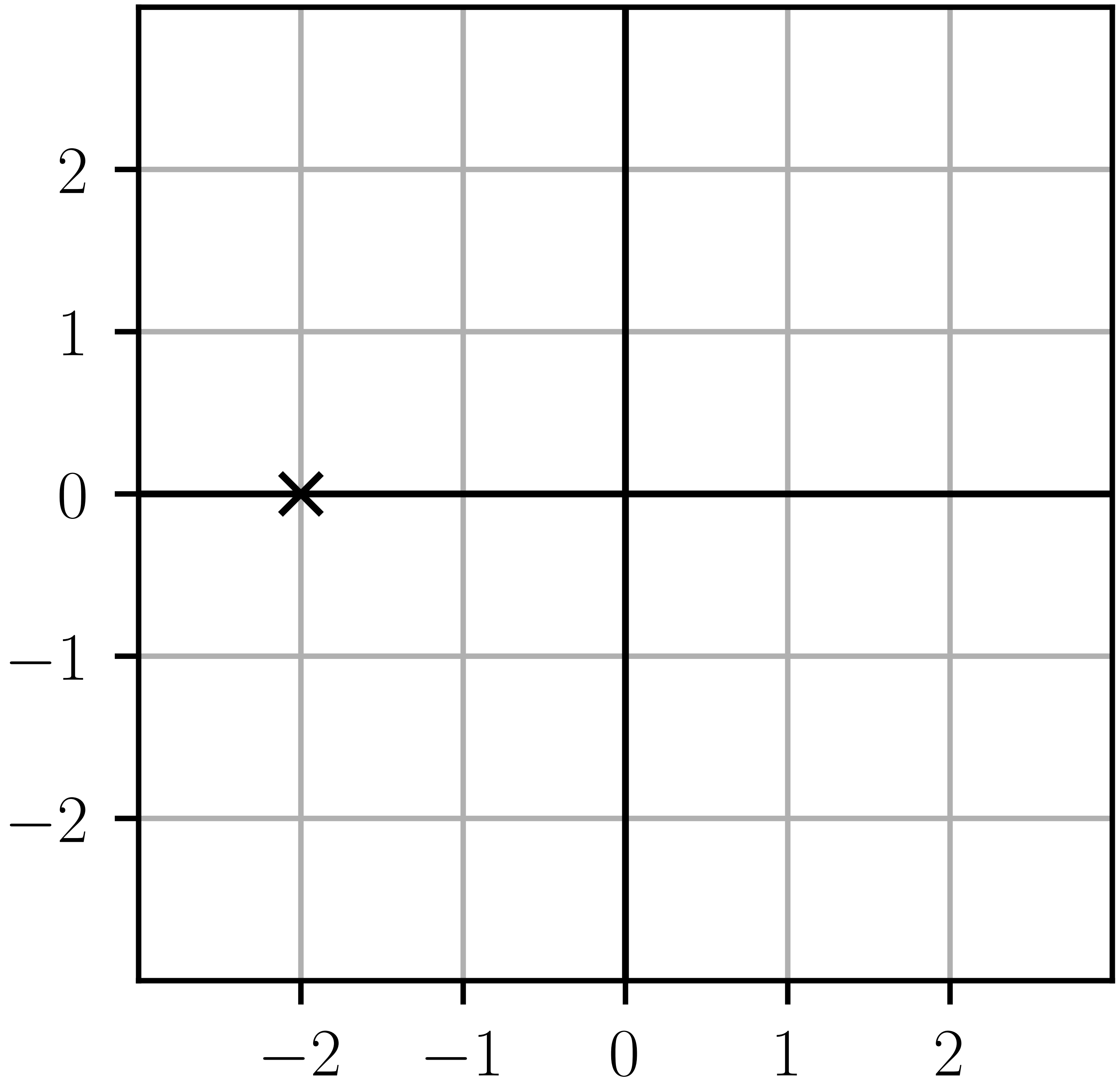


```
a = -2.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```


$a = -2.0$
10
8
6
4
2
0
$x(t)$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t

```python
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

$a = -2.0$



🔬 ANALYSIS

The origin is globally asymptotically stable when

$$a < 0.0$$

i.e. a **is in the open left-hand plane.**

Let the **time constant** τ be

$$\tau := 1/|a|.$$

When the system is asymptotically stable,

$$x(t) = e^{-t/\tau} x_0.$$

QUANTITATIVE CONVERGENCE

τ controls the speed of convergence to the origin:

time t	**distance to the origin** $\lvert x(t)\rvert$
0	$\lvert x(0)\rvert$
τ	$\simeq (1/3)\lvert x(0)\rvert$
3τ	$\simeq (5/100)\lvert x(0)\rvert$
$\vdots$	$\vdots$
$+\infty$	0

VECTOR CASE, DIAGONAL, REAL-VALUED

$$\dot{x}_1 = a_1 x_1,\ x_1(0) = x_{10}$$

$$\dot{x}_2 = a_2 x_2,\ x_2(0) = x_{20}$$

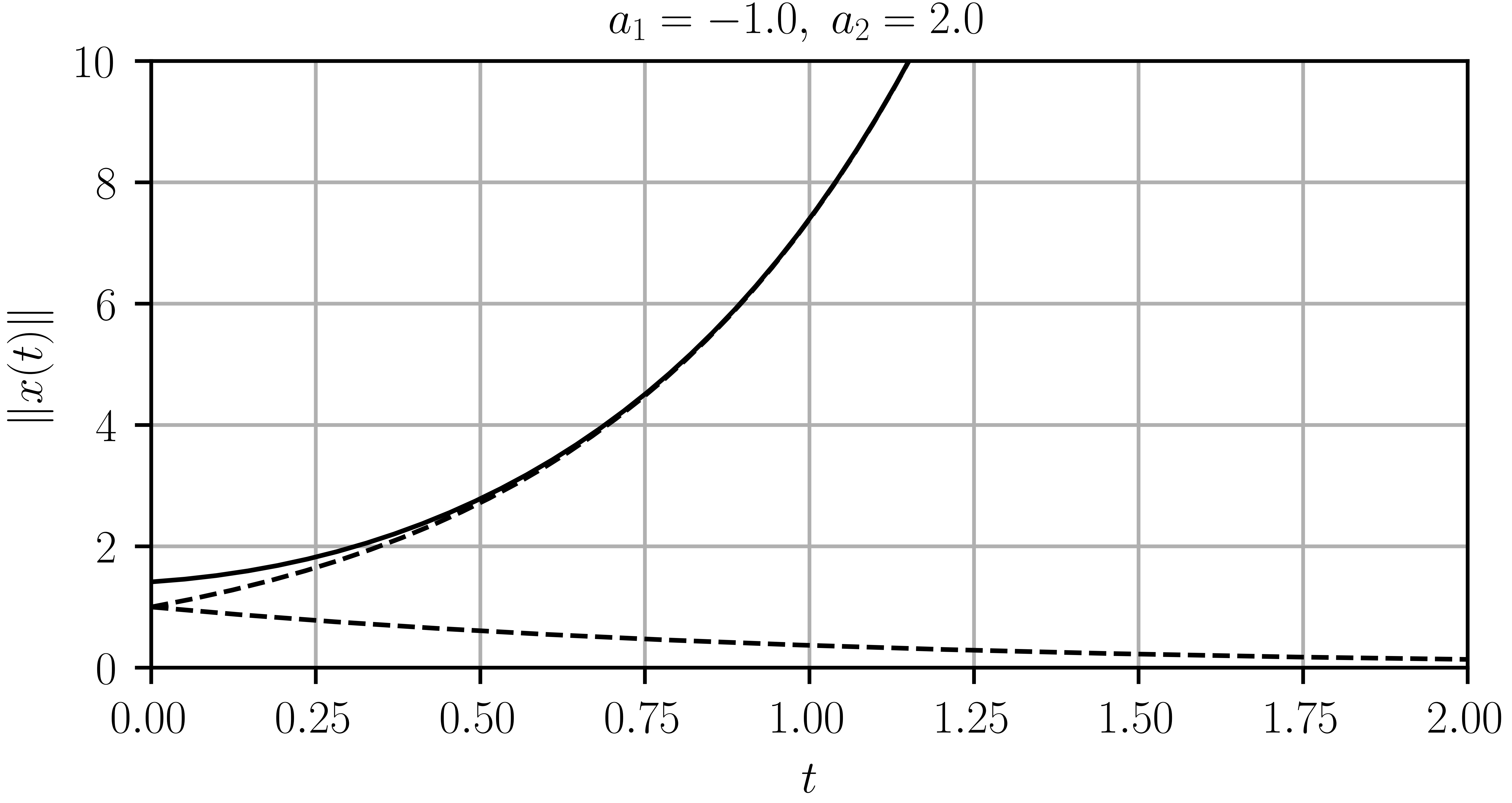
i.e.

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

🔓 **Solution:** by linearity

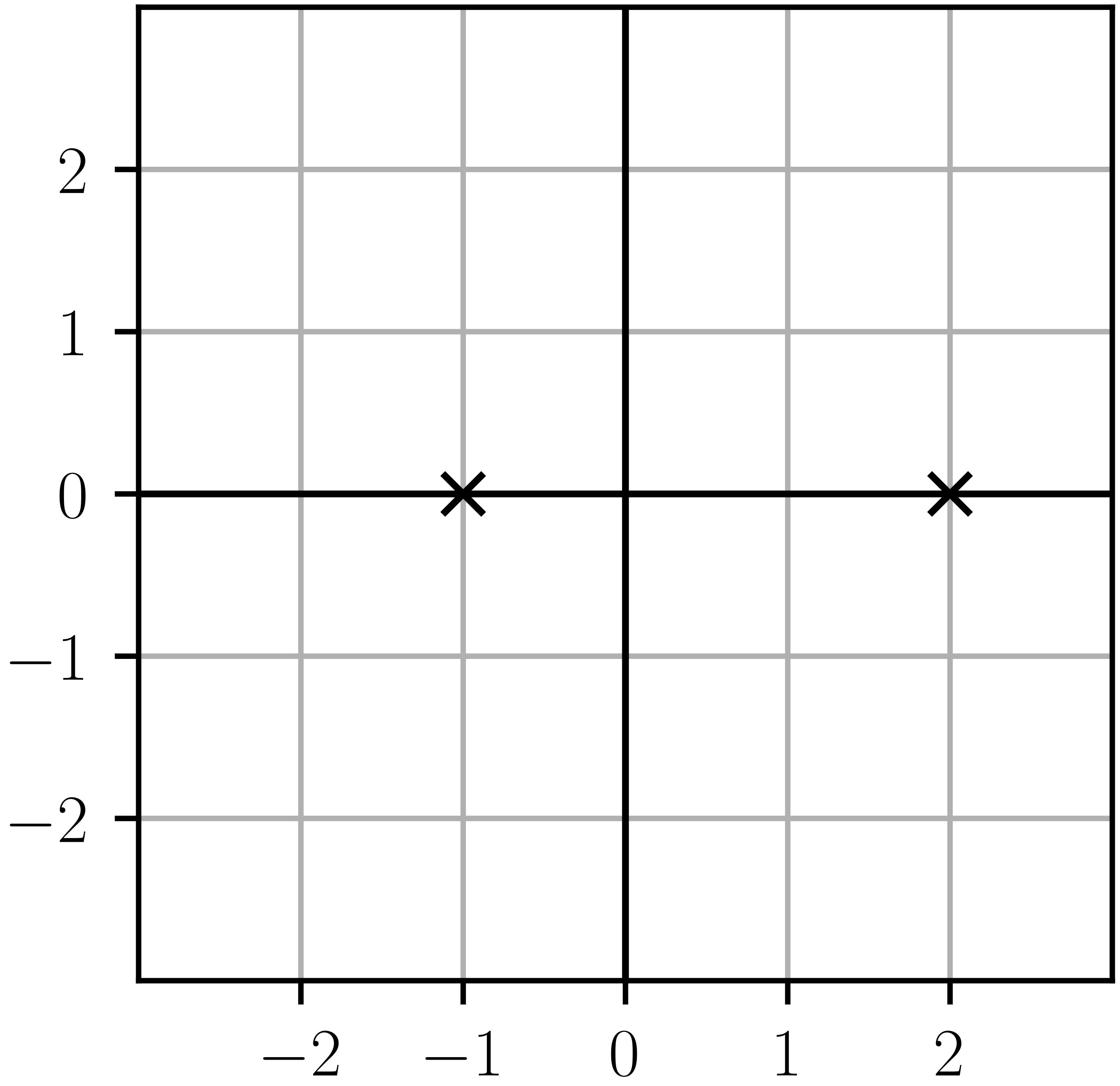
$$x(t) = e^{a_1 t} \begin{bmatrix} x_{10} \\ 0 \end{bmatrix} + e^{a_2 t} \begin{bmatrix} 0 \\ x_{20} \end{bmatrix}$$

```
a1 = -1.0; a2 = 2.0; x10 = x20 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
x1 = exp(a1*t)*x10; x2 = exp(a2*t)*x20
xn = sqrt(x1**2 + x2**2)
plot(t, xn , "k")
plot(t, x1, "k--")
plot(t, x2 , "k--")
xlabel("$t$"); ylabel("$\|x(t)\|$"); title(f"$a_1={a1}, \; a_2
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

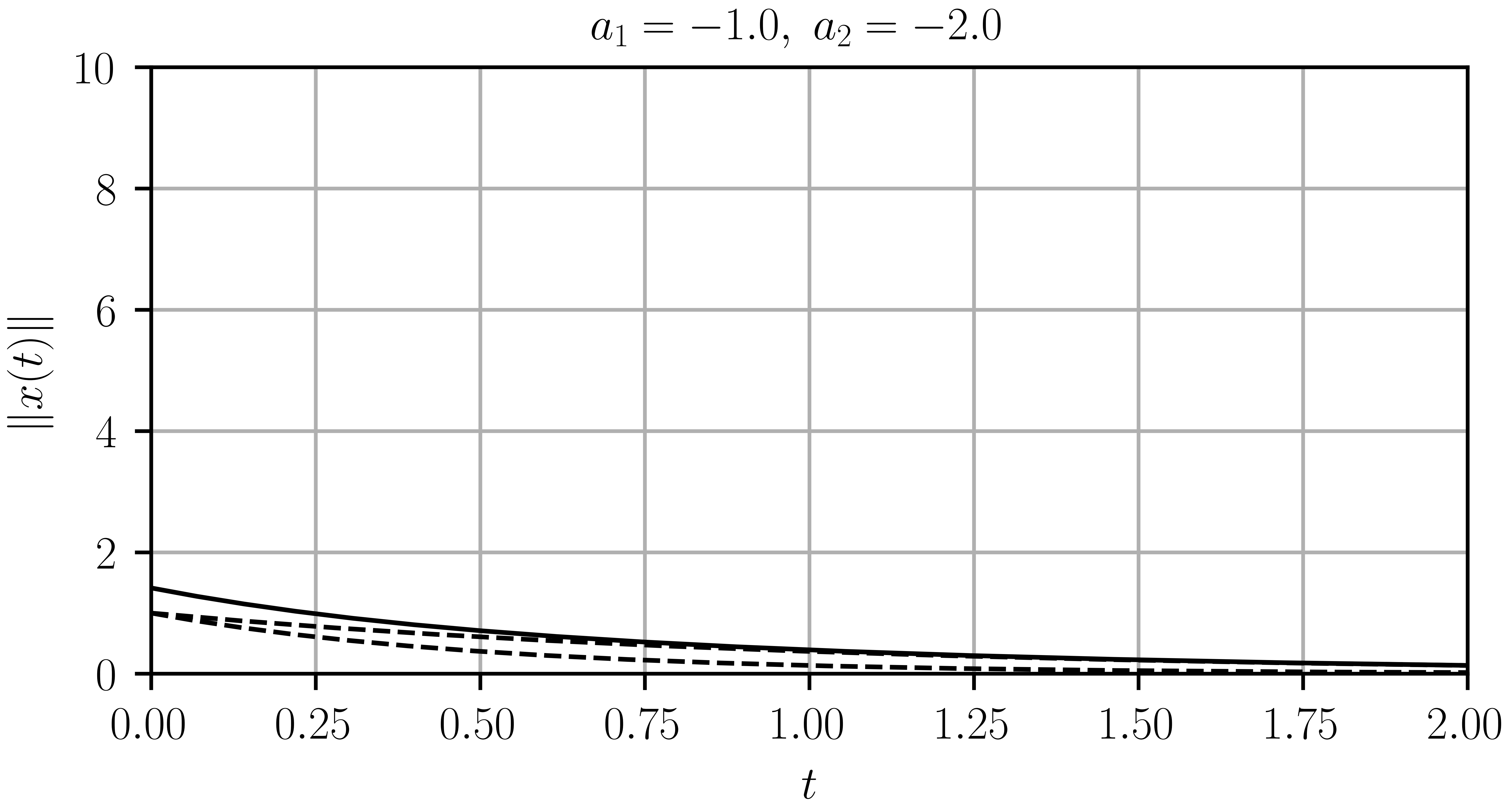

$a_1 = -1.0,\ a_2 = 2.0$
10
8
6
4
2
0
$\|x(t)\|$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t

```python
figure()
plot(real(a1), imag(a1), "x", color="k")
plot(real(a2), imag(a2), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a_1={a1}, \; a_2={a2}$")
grid(True)
```

$$a_1 = -1.0,\ a_2 = 2.0$$



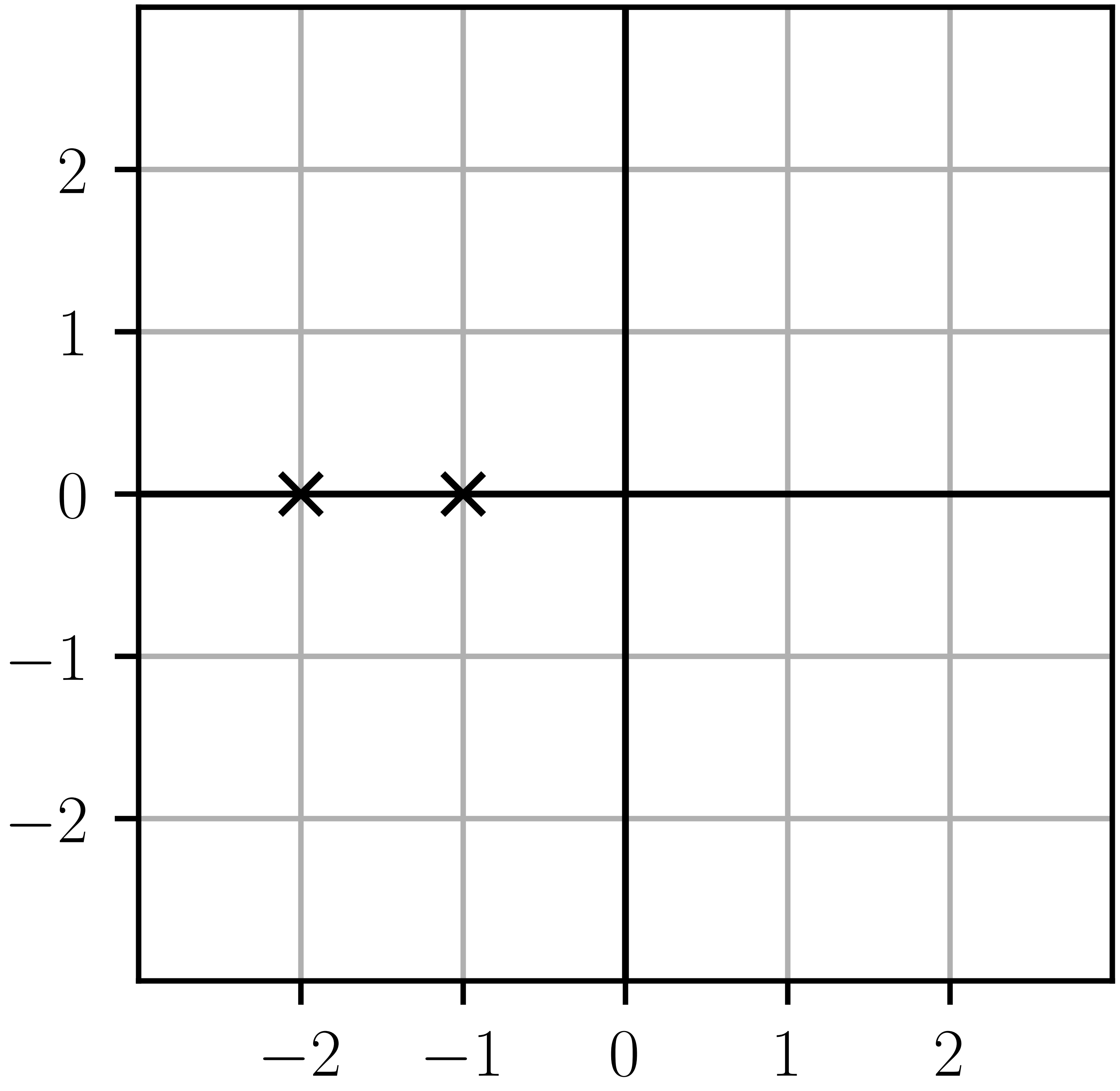
```
a1 = -1.0; a2 = -2.0; x10 = x20 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
x1 = exp(a1*t)*x10; x2 = exp(a2*t)*x20
xn = sqrt(x1**2 + x2**2)
plot(t, xn , "k")
plot(t, x1, "k--")
plot(t, x2 , "k--")
xlabel("$t$"); ylabel("$\|x(t)\|$"); title(f"$a_1={a1}, \; a_2
grid(); axis([0.0, 2.0, 0.0, 10.0])
```



$a_1 = -1.0,\ a_2 = -2.0$
10
8
6
4
2
0
$\|x(t)\|$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t

```python
figure()
plot(real(a1), imag(a1), "x", color="k")
plot(real(a2), imag(a2), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a_1={a1}, \; a_2={a2}$")
grid(True)
```

$a_1 = -1.0, \ a_2 = -2.0$



🔬 ANALYSIS

- 💎 The rightmost a_i determines the asymptotic behavior,
- 💎 The origin is globally asymptotically stable if and only if

  every a_i is in the open left-hand plane.

SCALAR CASE, COMPLEX-VALUED

$$\dot{x} = ax$$

$a \in \mathbb{C}, x(0) = x_0 \in \mathbb{C}.$

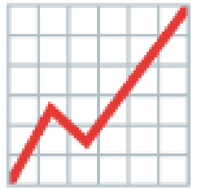
🔓 **Solution:** formally, the same old solution

$$x(t) = e^{at} x_0$$

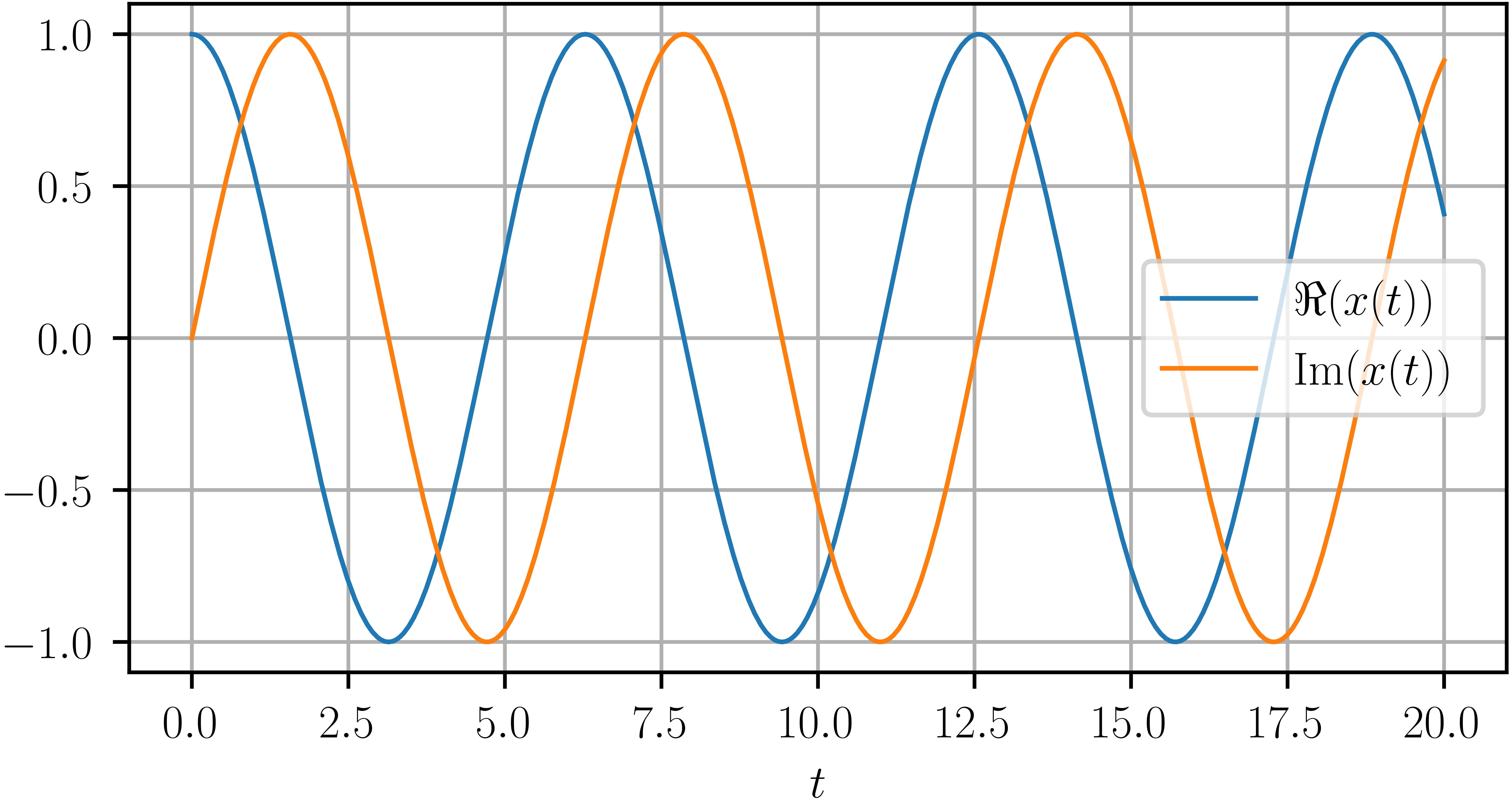
But now, $x(t) \in \mathbb{C}$:

if $a = \sigma + i\omega$ and $x_0 = |x_0| e^{i \angle x_0}$

$$|x(t)| = |x_0| e^{\sigma t} \; \text{and} \; \angle x(t) = \angle x_0 + \omega t.$$

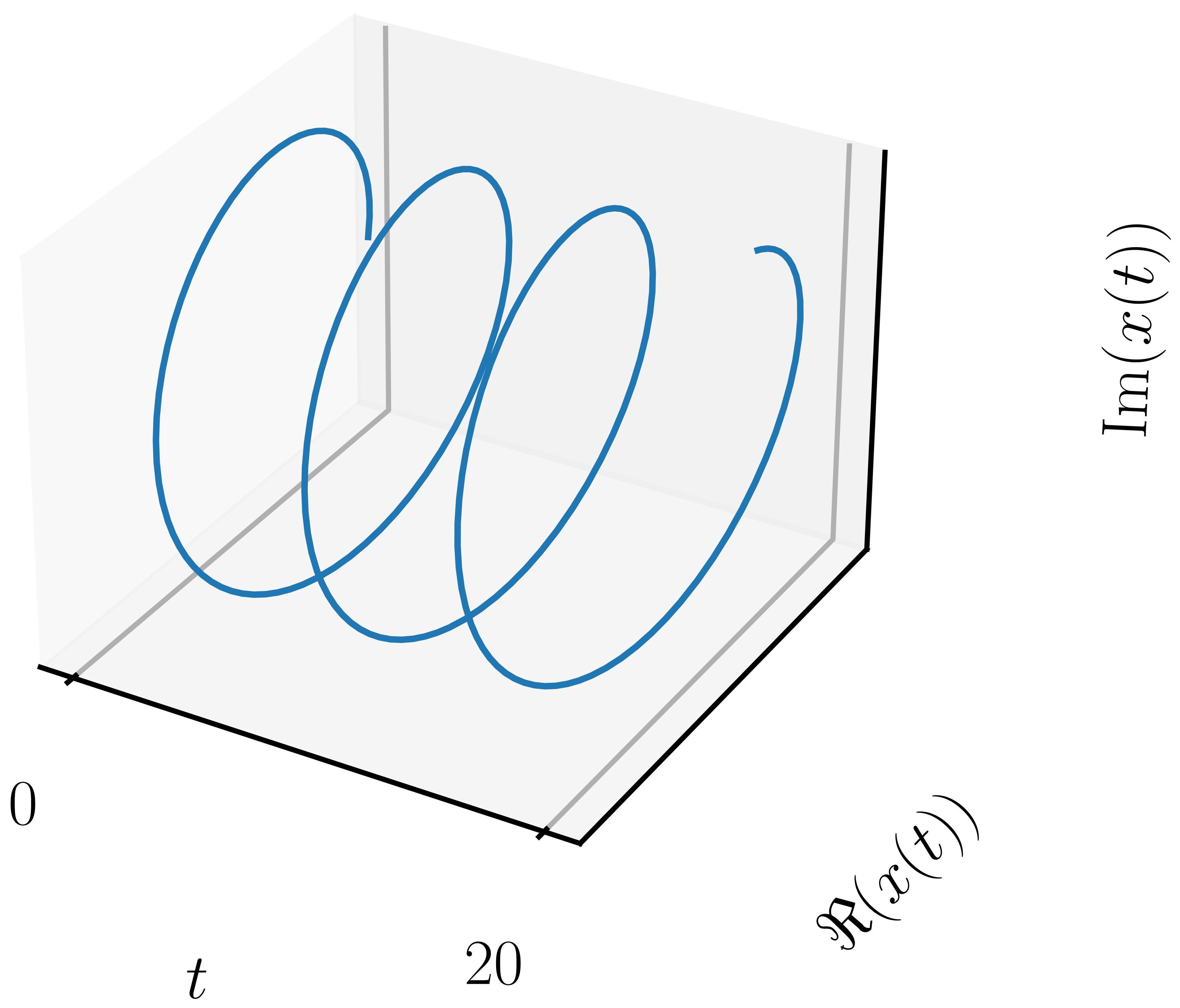


```
a = 1.0j; x0=1.0
figure()
t = linspace(0.0, 20.0, 1000)
plot(t, real(exp(a*t)*x0), label="$\Re(x(t))$")
plot(t, imag(exp(a*t)*x0), label="$\mathrm{Im}(x(t))$")
xlabel("$t$")
legend(); grid()
```



1.0
0.5
0.0
−0.5
−1.0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t
$\Re(x(t))$
$\mathrm{Im}(x(t))$

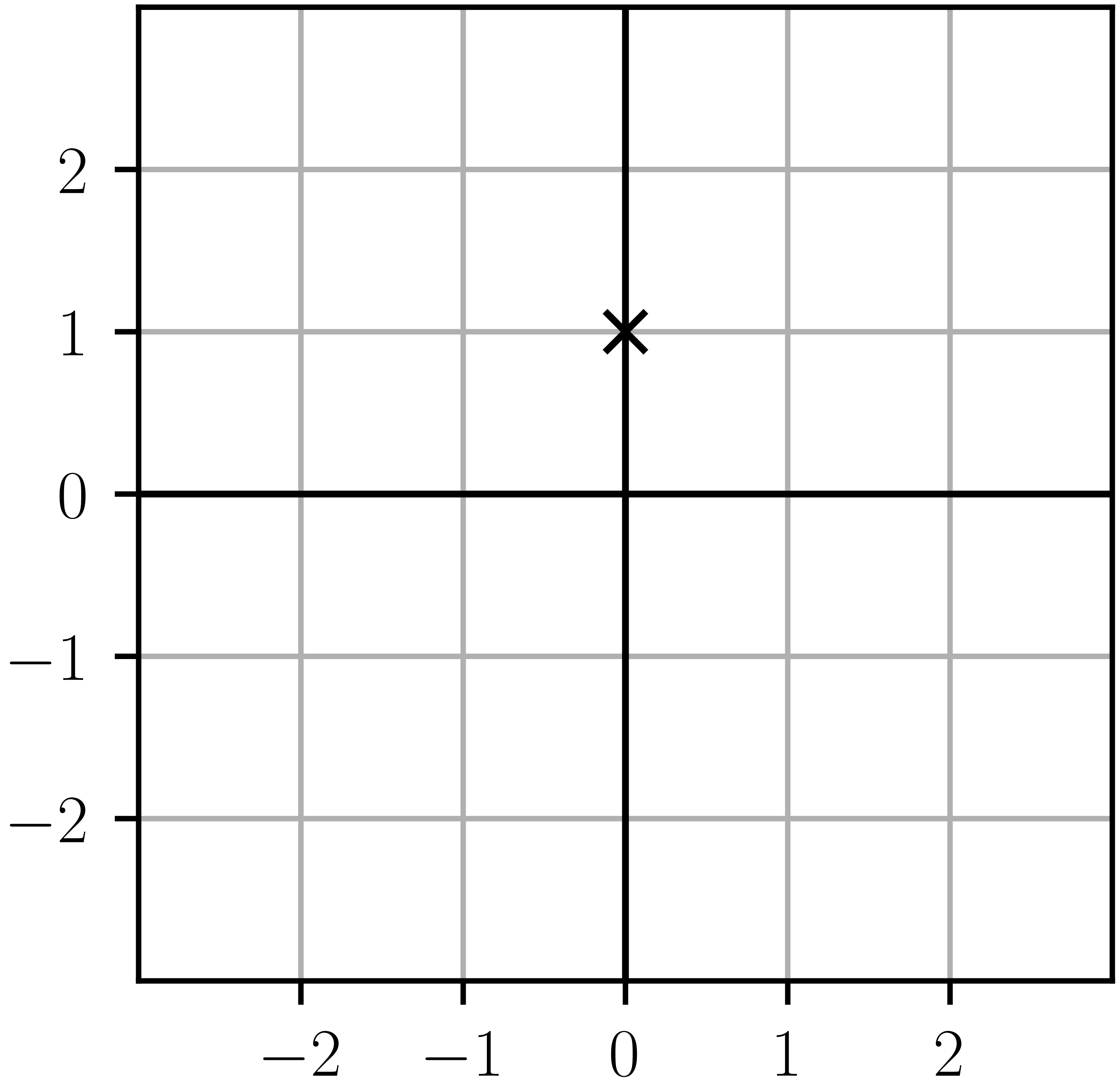
```python
fig = figure()
ax = fig.add_subplot(111, projection="3d")
zticks = ax.set_zticks
ax.plot(t, real(exp(a*t)*x0), imag(exp(a*t)*x0))
xticks([0.0, 20.0]); yticks([]); zticks([])
ax.set_xlabel("$t$")
ax.set_ylabel("$\Re(x(t))$")
ax.set_zlabel("$\mathrm{Im}(x(t))$")
```



0
t
20
$\Re(x(t))$
$\mathrm{Im}(x(t))$

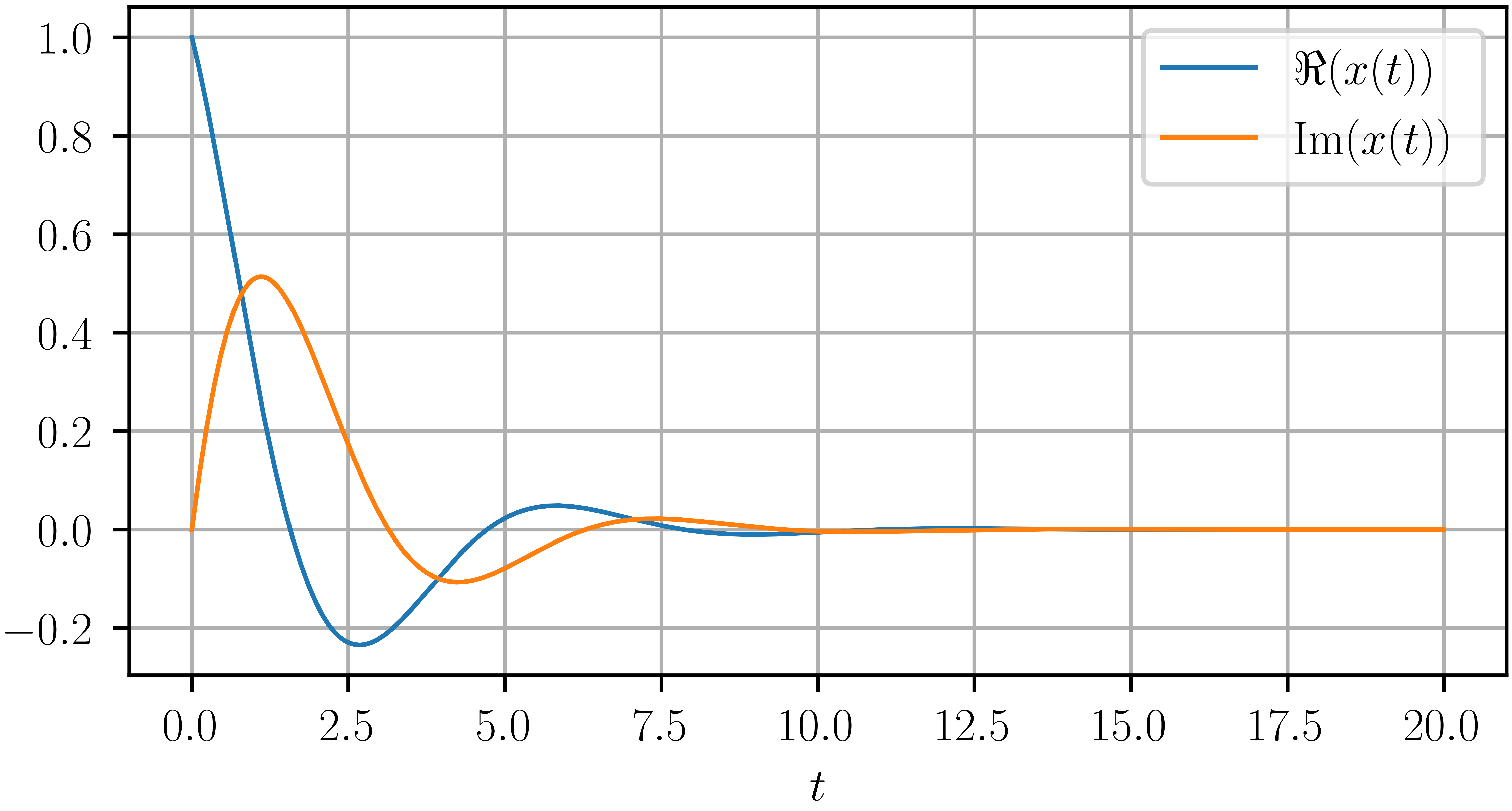
```python
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

$a = 1j$

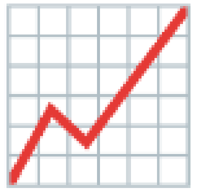


📈

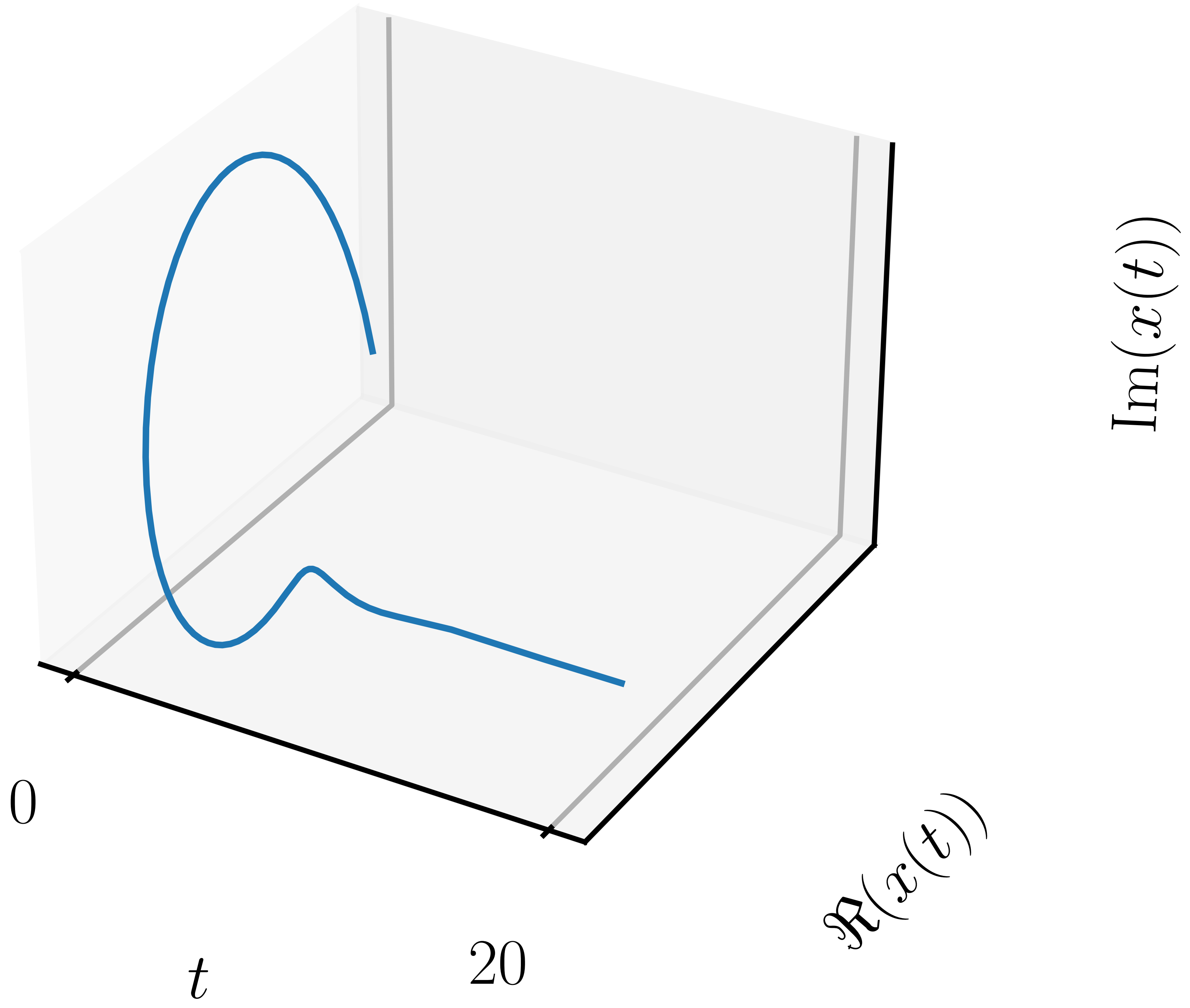
```python
a = -0.5 + 1.0j; x0=1.0
figure()
t = linspace(0.0, 20.0, 1000)
plot(t, real(exp(a*t)*x0), label="$\Re(x(t))$")
plot(t, imag(exp(a*t)*x0), label="$\mathrm{Im}(x(t))$")
xlabel("$t$")
legend(); grid()
```

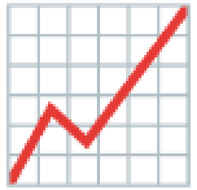


$\Re(x(t))$
$\mathrm{Im}(x(t))$
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t



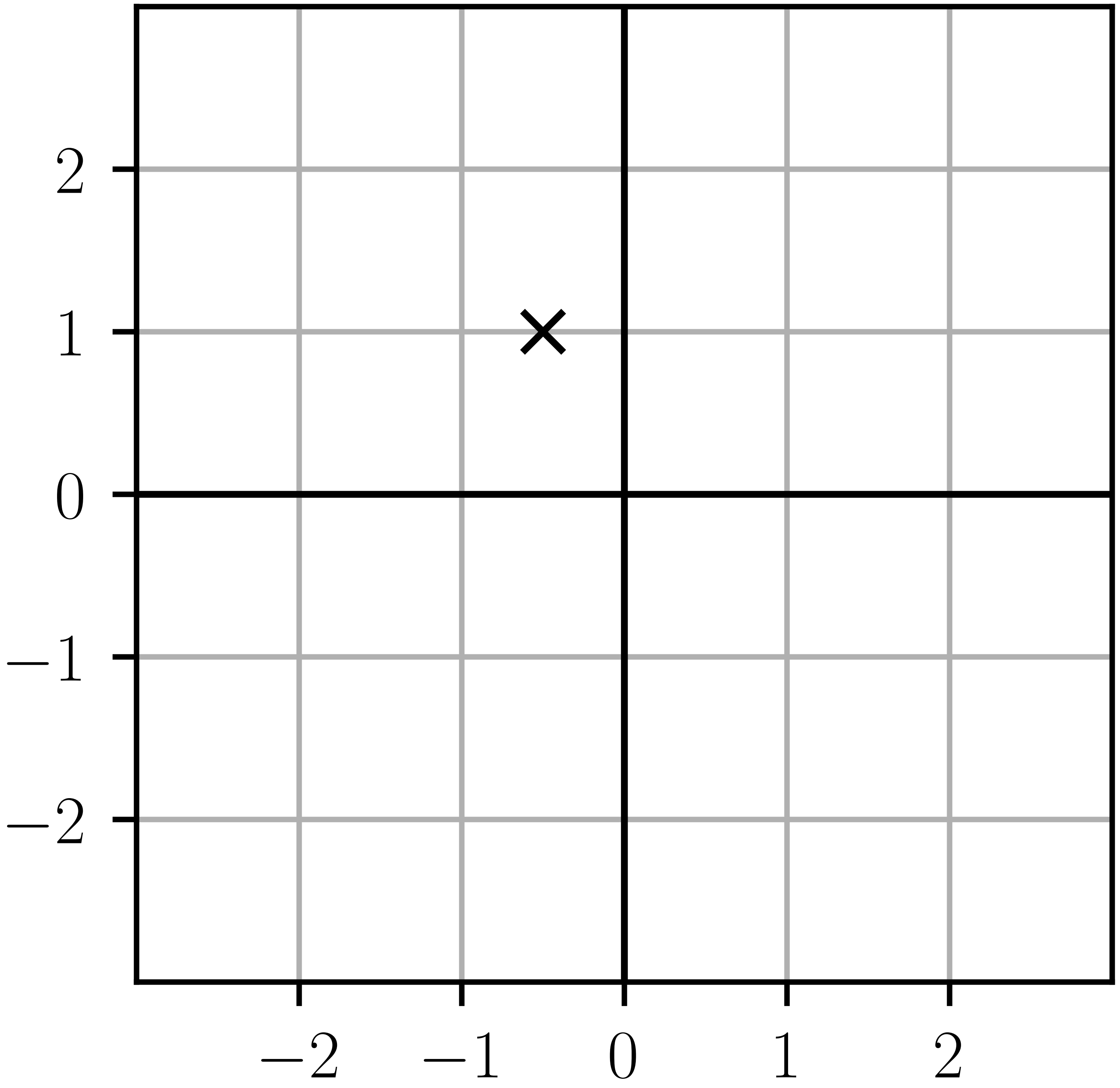
```python
fig = figure()
ax = fig.add_subplot(111, projection="3d")
zticks = ax.set_zticks
ax.plot(t, real(exp(a*t)*x0), imag(exp(a*t)*x0))
xticks([0.0, 20.0]); yticks([]); zticks([])
ax.set_xlabel("$t$")
ax.set_ylabel("$\Re(x(t))$")
ax.set_zlabel("$\mathrm{Im}(x(t))$")
```


$\mathrm{Im}(x(t))$
$\Re(x(t))$
0
t
20



```
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$")
grid(True)
```

$$a = (-0.5 + 1j)$$



🔬 ANALYSIS

- 💎 The origin is globally asymptotically stable iff a **is in the open left-hand plane:** $\Re(a) < 0$.
- 💎 If $a =: \sigma + i\omega$,
  - 🏷️ $\tau = 1/|\sigma|$ is the **time constant.**
  - 🏷️ ω the **rotational frequency** of the oscillations.

🏷️ EXPONENTIAL MATRIX

If $M \in \mathbb{C}^{n \times n}$, its **exponential** is defined as:

$$e^M = \sum_{k=0}^{+\infty} \frac{M^k}{k!} \in \mathbb{C}^{n \times n}$$

⚠️

The exponential of a matrix M is **not** the matrix with elements $e^{M_{ij}}$ (the elementwise exponential).

- 🐍 elementwise exponential: **`exp`** (`numpy` module),
- 🐍 exponential: **`expm`** (`scipy.linalg` module).

🧩 EXPONENTIAL MATRIX

Let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

1. 🧮

Compute the exponential of M.

🗝️ **Hint:**

$$\cosh x := \frac{e^x + e^{-x}}{2}, \; \sinh x := \frac{e^x - e^{-x}}{2}.$$

2. 🐍 💻 🔬

Compute numerically:

- `exp(M)` (`numpy`)
- `expm(M)` (`scipy.linalg`)

and check the results consistency.

🔓 EXPONENTIAL MATRIX

1. 🔓

We have

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ M^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and hence for any $j \in \mathbb{N}$,

$$M^{2j+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ M^{2j} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$
\begin{aligned}
e^M &= \sum_{k=0}^{+\infty} \frac{M^k}{k!} \\
&= \left(\sum_{j=0}^{+\infty} \frac{1}{(2j)!} \right) I + \left(\sum_{j=0}^{+\infty} \frac{1}{(2j+1)!} \right) M \\
&= \left(\sum_{k=0}^{+\infty} \frac{1^k + (-1)^k}{2(k!)} \right) I + \left(\sum_{k=0}^{+\infty} \frac{1^k - (-1)^k}{2(k!)} \right) M \\
&= (\cosh 1) I + (\sinh 1) M
\end{aligned}
$$

Thus,

$$e^M = \begin{bmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{bmatrix}.$$

2. 🔓

```
>>> M = [[0.0, 1.0], [1.0, 0.0]]
```

```
>>> exp(M)
array([[1.        , 2.71828183],
       [2.71828183, 1.        ]])
```

```
>>> expm(M)
array([[1.54308063, 1.17520119],
       [1.17520119, 1.54308063]])
```

These results are consistent:

```
>>> array([[exp(0.0), exp(1.0)],
...        [exp(1.0), exp(0.0)]])
array([[1.        , 2.71828183],
       [2.71828183, 1.        ]])
```

```
>>> array([[cosh(1.0), sinh(1.0)],
...        [sinh(1.0), cosh(1.0)]])
array([[1.54308063, 1.17520119],
       [1.17520119, 1.54308063]])
```

Note that

$$
\begin{aligned}
\frac{d}{dt} e^{At} &= \frac{d}{dt} \sum_{n=0}^{+\infty} \frac{A^n}{n!} t^n \\
&= \sum_{n=1}^{+\infty} \frac{A^n}{(n-1)!} t^{n-1} \\
&= A \sum_{n=1}^{+\infty} \frac{A^{n-1}}{(n-1)!} t^{n-1} = A e^{At}
\end{aligned}
$$

Thus, for any $A \in \mathbb{C}^{n\times n}$ and $x_0 \in \mathbb{C}^n$,

$$\frac{d}{dt}(e^{At}x_0) = A(e^{At}x_0)$$

💎 INTERNAL DYNAMICS

The solution of

$$\dot{x} = Ax \;\text{ and }\; x(0) = x_0$$

is

$$x(t) = e^{At} x_0.$$

🧩 G.A.S. $\Leftrightarrow$ L.A.

📝 For any dynamical system, if the origin is a globally asymptotically stable equilibrium, then it is a locally attractive equilibrium.

💎 **For linear systems**, the converse result also holds.

🚀 Let’s prove this!

1. 🧠

Show that for any linear system $\dot{x} = Ax$, if the origin is locally attractive, then it is also globally attractive.

2. 🧠

Show that linear system $\dot{x} = Ax$, if the origin is globally attractive, then it is also globally asymptotically stable.

🗝️ **Hint:** Consider the solutions $e_k(t) := e^{At} e_k$ associated to $e_k(0) = e_k$ where $(e_1, \dots, e_n)$ is the canonical basis of the state space.

🔓 G.A.S. ⇔ L.A.

1. 🔓

If the origin is locally attractive, then there is a $\varepsilon > 0$ such that for any $x_0 \in \mathbb{R}^n$ such that $\|x_0\| \leq \varepsilon$,

$$\lim_{t \to +\infty} e^{At} x_0 = 0.$$

Now, let any $x_0 \in \mathbb{R}^n$. Since the norm of $\varepsilon x_0 / \|x_0\|$ is ε, and by linearity of e^{At}, we obtain

$$
\begin{aligned}
\lim_{t \to +\infty} e^{At} x_0 &= \lim_{t \to +\infty} e^{At} \left(\frac{\|x_0\|}{\varepsilon} \varepsilon \frac{x_0}{\|x_0\|} \right) \\
&= \frac{\|x_0\|}{\varepsilon} \lim_{t \to +\infty} e^{At} \left(\varepsilon \frac{x_0}{\|x_0\|} \right) \\
&= 0.
\end{aligned}
$$

Thus the origin is globally attractive.

2. 🔓

Let X_0 be a bounded set of $\mathbb{R}^n$. Since

$$x_0 = \sum_{k=1}^{n} x_{0k} e_k,$$

the solution $x(t)$ of $\dot{x} = Ax, x(0) = x_0$ satisfies

$$x(t) = e^{At} x_0 = e^{At} \left(\sum_{k=1}^{n} x_{0k} e_k \right) = \sum_{k=1}^{n} x_{0k} e^{At} e_k.$$

$$
\begin{aligned}
\|x(t)\| &= \left\|\sum_{k=1}^{n} x_{0k} e^{At} e_k\right\| \\
&\leq \sum_{k=1}^{n} |x_{0k}| \left\|e^{At} e_k\right\| \\
&= \sum_{k=1}^{n} |x_{0k}| \, \|e_k(t)\| \\
&\leq \left(\sum_{k=1}^{n} |x_{0k}|\right) \max_{k=1,\ldots,n} \|e_k(t)\|
\end{aligned}
$$

Since X_0 is bounded, there is a $\alpha > 0$ such that for any $x_0 = (x_{01}, \ldots, x_{0n})$ in X_0,

$$\|x_0\|_1 := \sum_{k=1}^{n} |x_{0k}| \leq \alpha.$$

Since for every $k = 1, \ldots, n$, $\lim_{t \to +\infty} \|e_k(t)\| = 0$,

$$\lim_{t \to +\infty} \max_{k=1,\ldots,n} \|e_k(t)\| = 0.$$

Finally

$$\begin{aligned}\|x(t,x_0)\| &\leq \left(\sum_{k=1}^{n}|x_{0k}|\right)\max_{k=1,\ldots,n}\|e_k(t)\| \\ &\leq \alpha \max_{k=1,\ldots,n}\|e_k(t)\|\end{aligned}$$

Thus $\|x(t,x_0)\| \to 0$ when $t \to \infty$, uniformly w.r.t. $x_0 \in X_0$. In other words, the origin is globally asymptotically stable.

🏷️ EIGENVALUE & EIGENVECTOR

Let $A \in \mathbb{C}^n$. If $x \neq 0 \in \mathbb{C}^n$, $s \in \mathbb{C}$ and

$$Ax = sx$$

x is an **eigenvector of** A, s is an **eigenvalue** of A.

The **spectrum** of A is the set of its eigenvalues.
It is characterized by:

$$\sigma(A) := \{s \in \mathbb{C} \mid \det(sI - A) = 0\}.$$

🏷️ MODES & POLES

Consider the system $\dot{x} = Ax$.

- a **mode** of the system is an eigenvector of A,
- a **pole** of the system is an eigenvalue of A.

💎 STABILITY CRITERIA

Let $A \in \mathbb{C}^{n \times n}$.

The origin of $\dot{x} = Ax$ is globally asymptotically stable

$$\iff$$

all eigenvalues of A have a negative real part.

$$\iff$$

$$\max\{\Re s \mid s \in \sigma(A)\} < 0.$$

WHY DOES THIS CRITERIA WORK?

Assume that:

- **A is diagonalizable.**

(📝 very likely unless A has some special structure.)

Let $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$.

There is an invertible matrix $P \in \mathbb{C}^{n \times n}$ such that

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

Thus, if $y = P^{-1}x$, $\dot{x} = Ax$ is equivalent to

$$\left|\begin{array}{ccc} \dot{y}_1 & = & \lambda_1 y_1 \\ \dot{y}_2 & = & \lambda_2 y_2 \\ \vdots & = & \vdots \\ \dot{y}_n & = & \lambda_n y_n \end{array}\right.$$

The system is G.A.S. iff each component of the system is, which holds iff $\Re\lambda_i < 0$ for each i.

🧩 SPRING-MASS SYSTEM

Consider the scalar ODE

$$\ddot{x} + kx = 0, \quad \text{with } k > 0$$

1. 🧮

Represent this system as a first-order ODE.

2. 🧠 🧮

Is this system asymptotically stable?

3. 🧠 🧮

Do the solutions have oscillatory components?

Find the set of associated rotational frequencies.

4. 🧠 🧮

Same set of questions (1., 2., 3.) for

$$\ddot{x} + b\dot{x} + kx = 0$$

when $b > 0$.

🔓 SPRING-MASS SYSTEM

1. 🔓

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = A\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

2. 🔓

We have

$$\max\{\Re\, s \mid s \in \sigma(A)\} = 0,$$

hence the system is not globally asymptotically stable.

3. 🔓

Since

$$\det(sI - A) = \det \begin{pmatrix} s & -1 \\ k & s \end{pmatrix} = s^2 + k,$$

the spectrum of A is

$$\sigma(A) = \{s \in \mathbb{C} \mid \det(sI - A) = 0\} = \left\{i\sqrt{k}, -i\sqrt{k}\right\}.$$

The system poles are $\pm i\sqrt{k}$.

The general solution $x(t)$ can be decomposed as

$$x(t) = x_+ e^{i\sqrt{k}t} + x_- e^{-i\sqrt{k}t}.$$

Thus the components of $x(t)$ oscillate at the rotational frequency

$$\omega = \sqrt{k}.$$

4. 🔓

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = A\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\det(sI - A) = \det\begin{pmatrix} s & -1 \\ k & s+b \end{pmatrix} = s^2 + bs + k,$$

Let $\Delta := b^2 - 4k$. If $b \geq 2\sqrt{k}$, then $\Delta \geq 0$ and

$$\sigma(A) = \left\{ \frac{-b + \sqrt{\Delta}}{2}, \frac{-b - \sqrt{\Delta}}{2} \right\}.$$

Otherwise,

$$\sigma(A) = \left\{ \frac{-b + i\sqrt{-\Delta}}{2}, \frac{-b - i\sqrt{-\Delta}}{2} \right\}.$$

Thus, if $b \geq 2\sqrt{k}$,

$$\max\{\Re s \mid s \in \sigma(A)\} = \frac{-b + \sqrt{b^2 - 4k}}{2} < 0$$

and otherwise

$$\max\{\Re s \mid s \in \sigma(A)\} = -\frac{b}{2} < 0.$$

In each case, the system is globally asymptotically stable.

If $b \geq 2\sqrt{k}$, the poles are real-valued; the components of the solution do not oscillate.

If $0 < b < 2\sqrt{k}$, the imaginary part of the poles is

$$\pm\frac{\sqrt{4k - b^2}}{2} = \pm\sqrt{k - (b/2)^2},$$

thus the solution components oscillate at the rotational frequency

$$\omega = \sqrt{k - (b/2)^2}.$$

INTEGRATOR CHAIN

Consider the system

$$\dot{x} = Jx \;\; \text{with} \;\; J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

1. 🧮

Compute the solution $x(t)$ when

$$x(0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

2. 🦊 🧮

Compute the solution for an arbitrary $x(0)$

$$x(0) = \begin{bmatrix} x_1(0) \\ \vdots \\ \vdots \\ x_n(0) \end{bmatrix}.$$

3. 🧮

Same questions for the system

$$\dot{x} = (\lambda I + J)x$$

for some $\lambda \in \mathbb{C}$.

🗝️ **Hint:** Find the ODE satisfied by $y(t) := x(t)e^{-\lambda t}$.

4. 🧮

Is the system asymptotically stable ?

5. 🧠

Why does the stability analysis of this system matter ?

🔓 INTEGRATOR CHAIN

1. 🔓

Let $x = (x_1, \ldots, x_n)$.

The ODE $\dot{x} = Jx$ is equivalent to:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots \quad &\vdots \quad \vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= 0.
\end{aligned}
$$

When $x(0) = (0, \ldots, 0, 1)$,

- $\dot{x}_n = 0$ yields $x_n(t) = 1$, then
- $\dot{x}_{n-1} = x_n$ yields $x_{n-1}(t) = t$,
- …
- $\dot{x}_k = x_{k+1}$ yields

$$x_k(t) = \frac{t^{n-k}}{(n-k)!}.$$

To summarize:

$$x(t) = \begin{bmatrix} t^{n-1}/(n-1)! \\ \vdots \\ t^{n-1-k}/(n-1-k)! \\ \vdots \\ t \\ 1 \end{bmatrix}$$

2. 🔓

We note that

$$x(0) = x_1(0)\begin{bmatrix}1\\ \vdots\\ 0\\ 0\end{bmatrix} + \cdots + x_{n-1}(0)\begin{bmatrix}0\\ \vdots\\ 1\\ 0\end{bmatrix} + x_n(0)\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}.$$

Similarly to the previous question, we find that:

$$x(0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \ x(t) = \begin{bmatrix} t^{n-2}/(n-2)! \\ \vdots \\ t \\ 1 \\ 0 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} t^{n-3}/(n-3)! \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

And more generally, by linearity:

$$x(t) = \begin{bmatrix} x_1(0) + \cdots + x_{n-1}(0)\dfrac{t^{n-2}}{(n-2)!} + x_n(0)\dfrac{t^{n-1}}{(n-1)!} \\ \vdots \\ x_{n-2}(0) + x_{n-1}(0)t + x_n(0)\dfrac{t^2}{2} \\ x_{n-1}(0) + x_n(0)t \\ x_n(0) \end{bmatrix}$$

3. 🔓

If $\dot{x}(t) = (\lambda I + J)x(t)$ and $y(t) = x(t)e^{-\lambda t}$, then

$$\begin{aligned}
\dot{y}(t) &= \dot{x}(t)e^{-\lambda t} + x(t)(-\lambda e^{-\lambda t}) \\
&= (\lambda I + J)x(t)e^{-\lambda t} - \lambda I x(t)e^{-\lambda t} \\
&= Jx(t)e^{-\lambda t} \\
&= Jy(t).
\end{aligned}$$

Since $y(0) = x(0)e^{-\lambda 0} = x(0)$ we get

$$x(t) = \begin{bmatrix} x_1(0) + \cdots + x_n(0)\dfrac{t^{n-1}}{(n-1)!} \\ \vdots \\ x_{n-1}(0) + x_n(0)t \\ x_n(0) \end{bmatrix} e^{\lambda t}.$$

4. 🔓

The structure of $x(t)$ shows that

- If $\Re\lambda < 0$, then the system is asymptotically stable.
- If $\Re\lambda \geq 0$, then the system is not.

  For example when $x(0) = (1, 0, \ldots, 0)$, we have

$$x(t) = (1, 0, \ldots, 0).$$

5. 🔓

Every square complex matrix A, **even if it is not diagonalizable**, can be decomposed into a block-diagonal matrix where each block has the structure $\lambda I + J$.

Thus, the result of the previous question allows to prove the 💎 Stability Criteria in the general case.